Belirsiz Şekiller

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0.\infty$, $\infty-\infty$, 0^0, ∞^0, 1^∞ şeklindeki ifadeler belirsizdirler. (Aritmetik olarak bir anlamları yoktur.

Belirsizlik Tipi	Örnek
$\frac{0}{0}$	$\lim\limits_{x\to 0} \frac{\sin x}{x}$
$\frac{\infty}{\infty}$	$\lim\limits_{x\to \infty} \frac{e^x}{x^2}$
$0.\infty$	$\lim\limits_{x\to 0^+} x\ln\left(\frac{1}{x}\right)$
$\infty-\infty$	$\lim\limits_{x\to \frac{\pi}{2}^-} \left(\tan x-\frac{1}{\pi-2x}\right)$
0^0	$\lim\limits_{x\to 0^+} x^x$
∞^0	$\lim\limits_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x}$
1^∞	$\lim\limits_{x\to \infty} \left(1+\frac{1}{x}\right)^x$

NOT: 0^∞ belirsiz değildir.

L'hôpital Kuralı

$$\left.\begin{array}{lll} x\to a \text{ için} & f(x)\to 0 & g(x)\to 0 \\ x\to a \text{ için} & f(x)\to \pm\infty & g(x)\to \pm\infty \end{array}\right\} \text{ ise}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Burada a sonlu veya sonsuz olabilir.

$x\to a$ yerine $x\to a^+$ veya $x\to a^-$ de olabilir.

Örnek: $\lim\limits_{x\to\infty} \dfrac{\ln x}{2\sqrt{x}} = \dfrac{\ln\infty}{2\sqrt{\infty}} = \dfrac{\infty}{\infty}$

$\stackrel{L}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{2\,\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$

Örnek: $\lim\limits_{x\to 0} \underset{\frac{0}{0}}{\dfrac{2\sin x - \sin 2x}{2e^x - 2 - 2x - x^2}} \stackrel{L}{=} \underset{\frac{0}{0}}{\dfrac{2\cos x - 2\cos 2x}{2e^x - 2 - 2x}} \stackrel{L}{=} \underset{\frac{0}{0}}{\dfrac{-2\sin x + 4\sin 2x}{2e^x - 2}}$

$\stackrel{L}{=} \dfrac{-2\cos x + 8\cos 2x}{2e^x}$

$= \dfrac{6}{2} = 3$

L' hôpital Kuralının İspatı

$\lim\limits_{x\to 1} \dfrac{x^{10}-1}{x^2-1} = ?$ $\quad \lim\limits_{x\to 1} \dfrac{x^{10}-1/x-1}{x^2-1/x-1}$

$f(x) = x^{10}-1 \quad f(1) = 0 \; , \quad \dfrac{\overset{x^{10}-1}{f(x)} - \overset{0}{f(1)}}{x-1} = f'(1)$

$g(x) = x^2 - 1$

$\lim\limits_{x\to 1} \dfrac{f'(1)}{g'(1)} = \dfrac{10x^9}{2x}\Big|_{x=1} = 5$

Genel olarak $f(a) = g(a) = 0$ ve $g'(a) \neq 0$ ise,

$\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{\frac{f(x)}{x-a}}{\frac{g(x)}{x-a}} = \lim\limits_{x\to a} \dfrac{\frac{f(x)-f(a)}{x-a}}{\frac{f(x)-f(a)}{x-a}} = \lim\limits_{x\to a} \dfrac{f'(a)}{g'(a)}$

Örnek: 1) $a>0 \quad \lim\limits_{x\to\infty} \dfrac{e^{ax}}{x} \stackrel{L}{=} \lim\limits_{x\to\infty} \dfrac{a e^{ax}}{1} = \infty$ $\quad$ (e^{ax} daha hızlı)

2) $\lim\limits_{x\to\infty} \dfrac{e^{ax}}{x^{10}} \stackrel{L}{=} \lim\limits_{x\to\infty} \dfrac{a e^{ax}}{10x^9} \stackrel{L}{=} \lim\limits_{x\to\infty} \dfrac{a^2 e^{ax}}{90x^8} \stackrel{L}{=} \cdots$

$= \lim\limits_{x\to\infty} \dfrac{a^{10} e^{ax}}{10!} = \infty$

3) $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^{1/3}} = 0$ (Neden?)

$\lim\limits_{x\to 0} x^x = ?$ $x>0$

$\lim\limits_{x\to 0} x^x = \lim\limits_{x\to 0} e^{x\ln x} = e^{\lim\limits_{x\to 0} x\ln x}$

$\lim\limits_{x\to 0} x\ln x = \lim\limits_{x\to 0} \dfrac{\ln x}{\frac{1}{x}} \overset{L}{=} \lim\limits_{x\to 0} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0} -x = 0$

Örnek: $\lim\limits_{x\to 1} \dfrac{x^3-3x+2}{1-x^2-2\sin(1-x)} = ?$

$\lim\limits_{x\to 1} x^3-3x+2 = 0, \quad \lim\limits_{x\to 1} 1-x^2-2\sin(1-x) = 0 \rightarrow \dfrac{0}{0}$

$$\lim_{x\to 1} \frac{3x^2-3}{-2x-2\cos(1-x).(-1)}$$

$\lim\limits_{x\to 1} 3x^2-3 = 0, \quad \lim\limits_{x\to 1} \left(-2x-2\cos(1-x)(-1)\right) = 0$

$$\lim_{x\to 1} \frac{6x}{-2-2\sin(1-x)} = \frac{6}{-2} = -3$$

$$\lim_{x\to\infty} a^x = \begin{cases} \infty, & a>1 \\ 0, & 0<a<1 \\ 1, & a=1 \end{cases}$$

Örnek: $\lim\limits_{x\to\infty} \dfrac{\ln(3^x+2^x)}{x} \rightarrow \dfrac{\infty}{\infty}$

$$\overset{L}{=} \lim_{x\to\infty} \frac{\dfrac{3^x\ln 3+2^x\ln 2}{3^x+2^x}}{1} = \lim_{x\to\infty} \frac{\cancel{3^x}\left(\ln 3+\left(\frac{2}{3}\right)^x \ln 2\right)}{\cancel{3^x}\left(1+\left(\frac{2}{3}\right)^x\right)}$$

$= \ln 3$

Örnek: $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x}-\dfrac{1}{\sin x}\right) \rightarrow \infty-\infty$

$$\lim_{x\to 0^+} \frac{\sin x - x}{x\sin x} \rightarrow \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0^+} \frac{\cos x-1}{\sin x+x\cos x} \rightarrow \frac{0}{0}$$

$$\overset{L}{=} \lim_{x\to 0^+} \frac{-\sin x}{\cos x+\cos x-x\sin x} = \lim_{x\to 0^+} \frac{-\sin x}{2\cos x-x\sin x} = \frac{0}{2} = 0$$

Örnek: $\lim\limits_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right) \to \infty.0$

$$\lim_{x\to\infty} \frac{1-\cos\frac{1}{x}}{\frac{1}{x^2}} \to \frac{0}{0}$$

$$\stackrel{L}{=} \lim_{x\to\infty} \frac{-\frac{1}{x^2}\sin\frac{1}{x}}{-\frac{2}{x^3}} \to \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{2.\frac{1}{x}} = \frac{1}{2}.1 = \frac{1}{2}$$

TERS TÜREVLER

Tanım. Bir I aralığındaki her x için $F'(x)=f(x)$ ise, I aralığındaki F fonksiyonuna f nin ters türevi denir.

$F(x)$ fonksiyonunu, türevi olan $F'(x)$ ten bulma işlemine ters türev alma denir.

Örnek: $f(x)=6x^2$ ters türevini hesaplayalım.

$$F'(x)=f(x)=6x^2 \Rightarrow F(x)=2x^3+1,\ 2x^3+2, \ldots, 2x^3+c$$

Eğer $F(x)$, I aralığında $f(x)$ in ters türevi ise, $f(x)$ in I üzerindeki en genel ters türevi $F(x)+c$ dir. Burada c keyfi sabittir.

Örnek: $f_1(x)=\sin 2x \Rightarrow F(x)=\frac{-\cos 2x}{2}+c$

$$f(x)=\frac{1}{2\sqrt{x}} \Rightarrow F(x)=\sqrt{x}+c$$

$$f(x)=e^x \Rightarrow F(x)=e^x+c$$

$$f(x)=\frac{1}{x} \Rightarrow F(x)=\ln x+c$$

BELİRSİZ İNTEGRALLER

Tanım. $f(x)$ fonksiyonunun tüm ters türevlerinin kümesine, f nin x e göre belirsiz integrali denir ve

$$\int \underbrace{f(x)}_{\text{İntegrand}} \cdot d\overbrace{x}^{\text{değişken}}$$

ile gösterilir.

Örnekler:

1) $\int 1\,dx = x + c$

2) $\int \frac{dx}{x} = \ln x + c$

3) $\int \frac{-dx}{x^2} = \frac{1}{x} + c$

4) $\int \cos x\,dx = \sin x + c$

5) $\int \sin x\,dx = -\cos x + c$

6) $\int x^2 dx = \frac{x^3}{3} + c$

Temel İntegrasyon Formülleri

1) $\int dx = x + c$

2) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

3) $\int e^x dx = e^x + c \quad , \quad \int e^{ax} dx = \frac{e^{ax}}{a} + c$

4) $\int a^x dx = \frac{a^x}{\ln a} + c$

5) $\int \sin x\,dx = -\cos x + c$

6) $\int \cos x\, dx = \sin x + c$

7) $\int \frac{dx}{\cos^2 x} = \int \sec^2 x\, dx = \tan x + c$

8) $\int \frac{dx}{\sin^2 x} = \int \operatorname{cosec}^2 x\, dx = -\cot x + c$

9) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$

10) $\int \frac{dx}{1+x^2} = \arctan x + c$

Örnek: 1) $\int \tan^2 x\, dx = \int (1+\tan^2 x - 1)\, dx$

$= \int (1+\tan^2 x)\, dx - \int 1 \cdot dx$

$= \tan x - x + c$

$\int (f(x) \mp g(x))\, dx = \int f(x)\, dx \pm \int g(x)$

$\int c f(x)\, dx = c \cdot \int f(x)\, dx$

2) $\int \sin^2 x\, dx + \int \cos^2 x\, dx$

$\int (\sin^2 x + \cos^2 x)\, dx = \int dx = x + C$

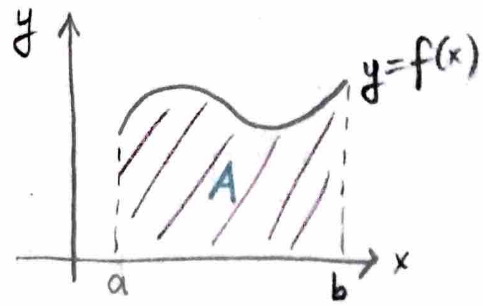
3) $\int (x^2+x)^2 dx = \int (x^4 + 2x^3 + x^2)\, dx$

$= \frac{x^5}{5} + \frac{2x^4}{4} + \frac{x^3}{3} + c$

4) $\int \frac{3x+2}{x^3}\, dx = 3\int \frac{1}{x^2} + 2\int \frac{1}{x^3}$

$= \frac{3x^{-1}}{-1} + \frac{2x^{-2}}{-2} + c = -\frac{3}{x} - \frac{1}{x^2} + c$

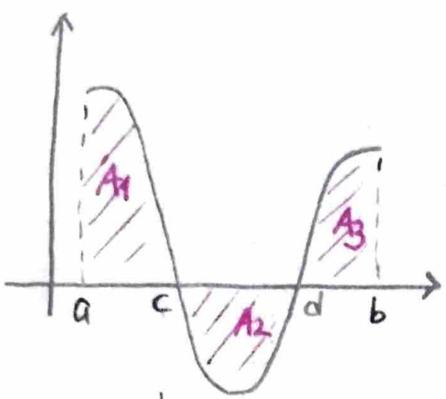
Belirli İntegral



integralin üst sınırı

$$A = \int_a^b f(x)\,dx \qquad (f(x) \geq 0)$$

integralin alt sınırı



$$\underbrace{\int_a^b f(x)\,dx}_{\text{toplam alan}} = A_1 - A_2 + A_3$$

$$= \int_a^c f(x)\,dx - \int_c^d f(x)\,dx + \int_d^b f(x)\,dx$$

$$\int_c^d f(x)\,dx < 0$$

Belirli İntegralin Özellikleri

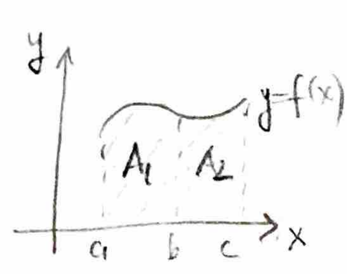
1) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2) $\displaystyle\int_a^a f(x)\,dx = 0$

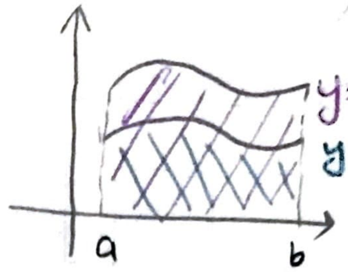
3) $\displaystyle\int_a^b k f(x)\,dx = k \cdot \int_a^b f(x)\,dx$ (k: sabit)

4) $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

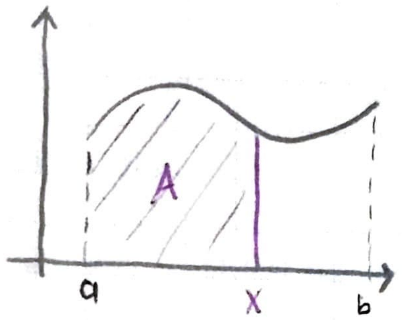
5) $\displaystyle\underbrace{\int_a^b f(x)\,dx}_{A_1} + \underbrace{\int_b^c f(x)\,dx}_{A_2} = \underbrace{\int_a^c f(x)\,dx}$



6) $[a,b]$ üzerinde $f(x) \geq g(x) \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$



Kalkülüsün Temel Teoremi



$$F(x) = \underbrace{\int_a^x f(t)dt}_{A}$$

$$\boxed{F'(x) = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)}$$

Örnek: $\frac{d}{dx}\left(\int_{\pi}^{x} \frac{\cos^2 t}{\ln(t-\sqrt{t})}dt\right) = \frac{\cos^2 x}{\ln(x-\sqrt{x})}$

$$y = \int_a^x (t^3+1)dt \Rightarrow \frac{dy}{dx} = x^3+1$$

Genel Form (Leibnitz Kuralı)

$$\frac{d}{dx}\left(\int_{u(x)}^{v(x)} f(t)dt\right) = f(v(x)).v'(x) - f(u(x)).u'(x)$$

Teorem. F, f nin ters türevi olmak üzere;

$$\int_a^b f(x)dx = F(b) - F(a)$$

Örnek: 1) $\displaystyle\int_1^4 \left(\frac{3\sqrt{x}}{2} - \frac{4}{x^2}\right) dx = \left(\frac{3}{2}\frac{x^{3/2}}{3/2} - \frac{4x^{-1}}{-1}\right)\Bigg|_1^4$

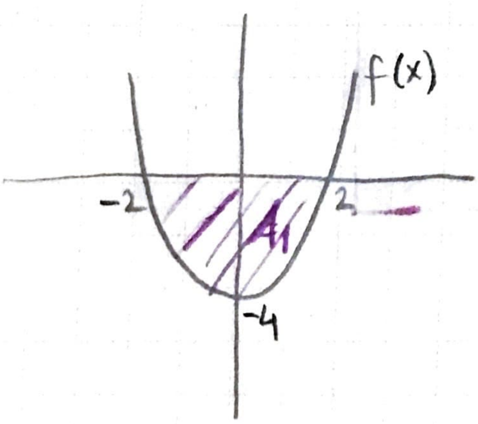
$$= 4^{3/2} + 4\cdot4^{-1} - (1^{3/2} + 4\cdot1^{-1}) = 4$$

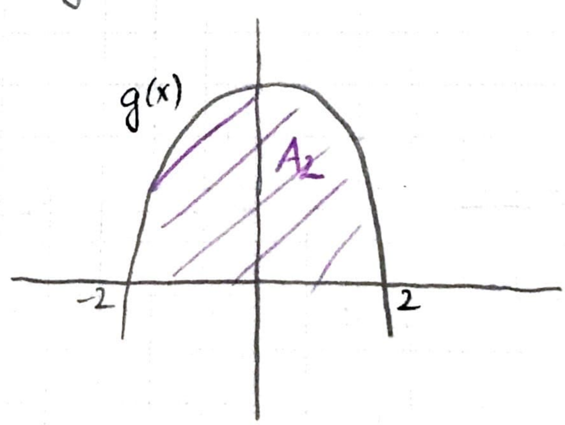
2) $\displaystyle\int_{-\pi/4}^{0} \sec x \tan x\, dx = \sec x \Big|_{-\pi/4}^{0} = \sec 0 - \sec\left(\frac{-\pi}{4}\right) = \frac{1}{\cos 0} - \frac{1}{\cos(\frac{-\pi}{4})} = 1-\sqrt{2}$

Örnek: $f(x) = x^2 - 4$ ve $g(x) = 4 - x^2$ fonksiyonları veriliyor.

Bu fonksiyonların a) $[-2,2]$ aralığında belirli integrallerini

b) $[-2,2]$ aralığında x-ekseni ile aralarında kalan bölgenin alanını bulunuz.





$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{2} (x^2-4)\,dx = \left(\frac{x^3}{3} - 4x\right)\Bigg|_{-2}^{2} = -\frac{32}{3}$$

$$\int_{-2}^{2} g(x)\,dx = \int_{-2}^{2} (4-x^2)\,dx = \left(4x - \frac{x^3}{3}\right)\Bigg|_{-2}^{2} = \frac{32}{3}$$

İntegrasyon Teknikleri

① Değişken Değiştirme Metodu

$$\int f(\underbrace{g(x)}_{u})\,\underbrace{g'(x)\,dx}_{du} = \int f(u)\,du$$

$$u = g(x) \Rightarrow du = g'(x)\,dx$$

Örnek:

$$\int \underbrace{(x^3+x)^5}_{u}\,\underbrace{(3x^2+1)\,dx}_{du} = \int u^5\,du = \frac{u^6}{6} + c = \frac{(x^3+x)^6}{6} + c$$

Örnek:

$$\int x^2 \sin \underbrace{x^3}_{u}\,dx = \int \frac{\sin u \cdot du}{3} = \frac{-\cos u}{3} + c$$

$$u = x^3 \Rightarrow du = 3x^2\,dx$$

$$\Rightarrow \frac{du}{3} = x^2\,dx$$

Örnek:

$$\int x\sqrt{\underbrace{2x+1}_{u}}\,dx \Rightarrow \int \frac{u-1}{2} \cdot \sqrt{u} \cdot \frac{du}{2} = \frac{1}{4}\int (u-1)\cdot\sqrt{u}\,du$$

$$u = 2x+1 \Rightarrow du = 2\,dx$$

$$x = \frac{u-1}{2}$$

$$= \frac{1}{4}\int \left(u\sqrt{u} - \sqrt{u}\right)du$$

$$= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + c$$

Örnek:

$$\int \cos(7\theta+3)\,d\theta = \int \frac{\cos u\,du}{7} = \frac{1}{7}\int \cos u\,du = \frac{1}{7}\sin u + c$$

$$u = 7\theta+3$$

$$du = 7\,d\theta$$

Örnek:

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{\sqrt[3]{u}} = \int u^{-1/3}\,du = \frac{u^{2/3}}{2/3} + c$$

$$u = z^2+1$$

$$du = 2z\,dz$$

Belirli İntegralde Değişken Değiştirme

$$\int_a^b f(\underbrace{g(x)}_{u})\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Örnek:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \int_0^2 \sqrt{u}\,du = \frac{2}{3}u^{3/2}+c\Big|_0^2 = \frac{4\sqrt{2}}{3}$$

$x^3+1=u$ $\quad 1^3+1$

$3x^2dx = du$ $\quad (-1)^3+1$

Simetrik Fonksiyonların Belirli İntegralleri

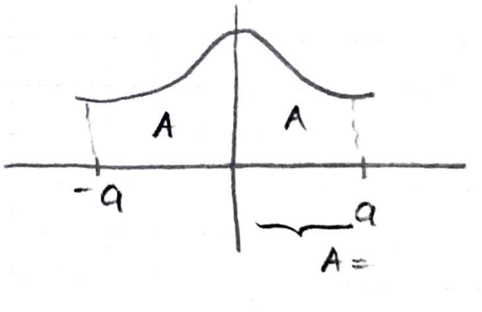
f, $[-a,a]$ simetrik aralığı üzerinde sürekli olsun.

i) f çift fonksiyon ise $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$(f(-x)=f(x))$

$\downarrow$

y eksenine göre simetrik

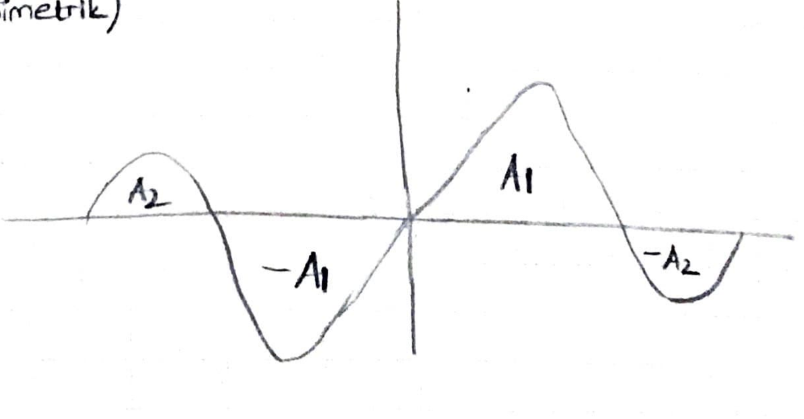


$A = \displaystyle\int_0^a f(x)\,dx$

ii) f tek fonksiyon ise $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$

$f(-x) = -f(x)$

(Orijine göre simetrik)



Örnek: $\displaystyle\int_{-2}^{2}(x^6+1)\,dx = 2\int_{0}^{2}(x^6+1)\,dx$ $f(x)=x^6+1$ çift

$$= 2\left(\frac{x^7}{7}+x\right)\Bigg|_0^2 = \frac{2^8}{7}+4$$

Örnek: $\displaystyle\int_{-\pi/2}^{\pi/2}\frac{x^2\sin x}{1+x^6}\,dx = 0$ $f(-x)=\dfrac{(-x)^2\sin(-x)}{1+(-x)^6}=-f(x)$ tek

Bazı Trigonometrik Fonksiyonların İntegralleri

1) $\displaystyle\int \sin x\,dx = -\cos x + c$

2) $\displaystyle\int \cos x\,dx = \sin x + c$

3) $\displaystyle\int \tan x\,dx = \int \frac{\overbrace{\sin x\,dx}^{-du}}{\underbrace{\cos x}_{u}} = -\int\frac{du}{u} = -\ln|u|+c = -\ln|\cos x|+c$

4) $\displaystyle\int \cot x\,dx = \int \frac{\overbrace{\cos x\,dx}^{du}}{\underbrace{\sin x}_{u}} = \int\frac{du}{u} = \ln|u|+c = \ln|\sin x|+c$

5) $\displaystyle\int \sec x\,dx = \int \frac{\overbrace{\sec x(\sec x+\tan x)\,dx}^{du}}{\underbrace{\sec x+\tan x}_{u}} = \int\frac{du}{u} = \ln|u|+c$

$= \ln|\sec x+\tan x|+c$

6) $\displaystyle\int \operatorname{cosec} x\,dx = -\ln|\operatorname{cosec} x+\cot x|+c$