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ANSWERS KEY

Hacettepe University MAT 123		Midterm Exam 28.11.2022	
Surname :	Instructor	:	Signature
Name :	Time	: 10.00	
ID :	Duration	: 90 dk	
5 questions, 2 pages 100 points			TOTAL:

Q1 (15 p.) Evaluate the following limits.

a) $\lim_{x\to\infty} \frac{\ln x}{x}$ (5)   b) $\lim_{x\to\infty} \left(\frac{\ln x}{x}\right)^{\frac{1}{x}}$ (10)

a) $\lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

b) $\lim_{x\to\infty} \left(\frac{\ln x}{x}\right)^{1/x}$ (0^0) $\Rightarrow \lim_{x\to\infty} e^{\frac{1}{x}\ln\left(\frac{\ln x}{x}\right)} = e^{\lim \frac{1}{x}\ln\left(\frac{\ln x}{x}\right)} = e^0 = 1$

$\lim_{x\to\infty} \frac{1}{x}\cdot\ln\left(\frac{\ln x}{x}\right) \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\ln(\ln(x)) - \ln(x)}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1/x}{\ln x} - \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x\ln x} - \frac{1}{x} = 0$

Q2 (20 p.) If $f(x)$ is continuous on R, find a, b and c, where $f(1) = c$ and

$$f(x) = \begin{cases} \frac{\sin(2x+2)}{x+1} & x < -1 \\ ax^2 + \frac{|x^2-1|}{\sqrt[3]{x-1}} & -1 \le x < 1 \\ (x-1)\sin\frac{1}{x-1} - b & x > 1. \end{cases}$$

$\lim_{x\to -1^-} \frac{\sin 2(x+1)}{x+1} = \lim_{x\to -1^-} 2\cdot\frac{\sin 2(x+1)}{2(x+1)} = 2$

$\lim_{x\to -1^+} ax^2 + \frac{|x^2-1|}{\sqrt[3]{x-1}} = a$ 4

$\Rightarrow \boxed{a=2}$

$\lim_{x\to 1^-} ax^2 + \frac{|x^2-1|}{\sqrt[3]{x-1}} = \lim_{x\to 1^-} ax^2 + \frac{|x-1||x+1|}{\sqrt[3]{x-1}} = \lim_{x\to 1^-} ax^2 + |x-1|^{2/3}|x+1| = a = 2$ 4

$\lim_{x\to 1^+} (x-1)\sin\frac{1}{x-1} - b = -b = 2 \Rightarrow \boxed{b=-2}$ 4

$-(x-1) \le (x-1)\sin\frac{1}{x-1} \le (x-1)$ by S.T. $\lim_{x\to 1^+}$

$\lim_{x\to 1} = 2 = f(1) = c \Rightarrow \boxed{c=2}$ 4

Q3 (20 p.) Write an equation for the tangent line to the curve $y = (e-1+\sec x)^{\sqrt{1+\sin x}}$ at $(0,e)$.

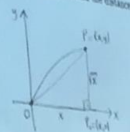
$\ln y = \sqrt{1+\sin x}\,\ln(e-1+\sec x)$ 10p

$\frac{y'}{y} = \frac{1}{2\sqrt{1+\sin x}}\cdot\cos x\,\ln(e-1+\sec x) + \sqrt{1+\sin x}\,\frac{\sec x\cdot\tan x}{e-1+\sec x}$

$x=0, y=e \Rightarrow \frac{y'}{e} = \frac{1}{2\sqrt{1}}\ln(e) + \sqrt{1+0}\,\frac{1\cdot 0}{e} \Rightarrow \boxed{y' = \frac{e}{2} \text{ is the slope}}$ 5p

$y - e = \frac{e}{2}(x-0) \Rightarrow \boxed{y = \frac{e}{2}x + e}$ is the tangent line. 5p

Q4 (15 p.) A point P is moving along the curve $x = y^2$ ... $O = (0,0)$, $P_0 = (x,0)$ and $P = (x,y)$. How fast ... is 4/3 m/min and the distance between O and P ...



$A(x) = \frac{x\sqrt{x}}{2} = \ldots$

$\frac{dA(x)}{dt} = \frac{3}{4}\ldots$

$= \frac{3}{4}\ldots$

Q5 (30 p.) Identifying the domain and all asym... curve is increasing and where it is decreasing... the curve

(2p)

Domain: $\mathbb{R} - \{-1\}$ Vertical...

$\lim_{x\to -1^+} e^{1/(x+1)} = 0$, $\lim_{x\to -1^-}$ ...

Horizontal asymptotes: y...

$\frac{dy}{dx} = -\frac{1}{(x+1)^2}e^{1/(x+1)} > 0$ for ...

$\frac{d^2y}{dx^2} = \ldots \frac{2x+1}{(x+1)^4}e^{1/(x+1)}$ (4p)

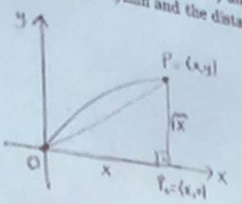
Critical points: No criti...

Singular points: No sin...

		$-1/2$	
y'	+		+
y''	+		
$y=f(x)$			

buraya yazın

Q4 (15 p.) A point P is moving along the curve $x = y^2$ in the first quadrant forming a triangle with vertices $O = (0,0)$, $P_0 = (x,0)$ and $P = (x,y)$. How fast is its area changing when the speed of the point P_0 is 4/3 m/min and the distance between O and P is $3\sqrt{10}$.



$A(x) = \frac{x \cdot \sqrt{x}}{2} = \frac{x^{3/2}}{2}$ (2p)

$\frac{dA(x)}{dt} = \frac{3}{4} \cdot x^{1/2} \cdot \frac{dx}{dt}$ (5p)

$= \frac{3}{4} \cdot \sqrt{9} \cdot \frac{4}{3}$

$= \boxed{3}$ (5p)

$x^2 + x = (3\sqrt{10})^2 = 90 \Rightarrow x^2 + x - 90 = 0$

$(x+10)(x-9) = 0 \Rightarrow \boxed{x=9}$

(3p)

Q5(30 p.) Identifying the domain and all asymptotes, investigating all critical points, intervals where the curve is increasing and where it is decreasing, points of inflections and the concavity of the curve, sketch the curve

$$y = e^{x/(x+1)}.$$

Domain: $\mathbb{R} - \{-1\}$ (2p) Vertical asymptotes: $x = -1$

$\lim_{x \to -1^+} e^{x/x+1} = 0$, $\lim_{x \to -1^-} e^{x/x+1} = \infty$ (4p)

Horizontal asymptotes: $y = e$. $\lim_{x \to \pm\infty} e^{x/x+1} = e$ (4p)

$\frac{dy}{dx} = \frac{1}{(x+1)^2} e^{x/x+1} > 0$ for all x (with $x \neq -1$) (4p)

$\frac{d^2y}{dx^2} = -\frac{2x+1}{(x+1)^4} \cdot e^{x/x+1}$ (4p)

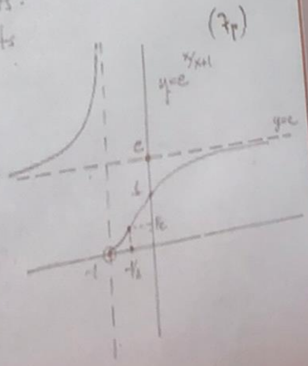
Critical points: No critical points.

Singular points: No singular points

(5p)

		-1		$-1/2$	
y'			+		+ 2p
y''			+		− 3p
$y=f(x)$			↗ ∪		↗ ∩

(7p)



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Q1 (15 p.) Evaluate the following limits.

8

a) $\lim_{x\to\infty} \frac{\ln x}{x}$

$\frac{\ln \infty}{\infty} = \frac{\infty}{\infty}$

3

l'hopital ✓

$\frac{(\ln x)'}{x'} = \frac{1/x}{1} = \frac{1}{x} = \frac{1}{\infty} = 0$

$-\frac{1}{x^2} \cdot -x^{-2}$

b) $\lim_{x\to\infty} \left(\frac{\ln x}{x}\right)^{\frac{1}{x}} = y$

$\ln y = \ln(\ln x / x) \cdot \frac{1}{x}$

5

$\ln y = \frac{\ln(\ln x) - \ln x}{x}$ l'hopital

$= \frac{\frac{1/x}{\ln x} - \frac{1}{x}}{1} = \frac{1 - \ln x}{x \cdot \ln x}$ l'hopital

$= \frac{-1/x}{1 \cdot 1/x} = \frac{-1 \cdot x}{x} = -1$

Q2 (20 p.) If $f(x)$ is continuous on $\mathbb{R}$, find a, b and c, where $f(1) = c$ and

$$f(x) = \begin{cases} \frac{\sin(2x+2)}{x+1} & x < -1 \\ ax^2 + \frac{|x^2-1|}{\sqrt[3]{x-1}} & -1 \le x < 1 \\ (x-1)\sin\frac{1}{x-1} - b & x > 1. \end{cases}$$

4

$\frac{\sin(-2+2)}{-1+1} = \frac{\sin 0}{0}$ lhopital $\frac{\sqrt[3]{x-1} \cdot ax^2 + |x^2-1|}{\sqrt[3]{x-1}} = \frac{\sqrt[3]{2} \cdot ax^2}{\sqrt[3]{2}} = 2$

$\frac{\cos(2x+2) \cdot 2}{1} = 2$

$a = 2$

$\lim_{x\to 1} ax^2 + \frac{|x^2-1|}{\sqrt[3]{x-1}} = \lim_{x\to 1} (x-1)\sin\frac{1}{x-1} - b$

not continuous at 1

Q3 (20 p.) Write an equation for the tangent line to the curve $y = (e - 1 + \sec x)^{\sqrt{1+\sin x}}$ at $(0, e)$.

find $= y = m(x - x_0) - y_0$

$y = 2(x - 0) - e$

$y = 2x - e$

5

$\ln y = \ln(e - 1 + \sec x) \cdot \sqrt{1+\sin x}$

$\ln y = \left(\frac{\ln(e-1+\sec x) \cdot (1+\sin x)}{\sqrt{1+\sin x}}\right) \frac{1 \cdot 1}{1}$

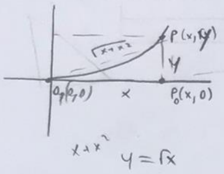
$\frac{(e-1+\sec x)'}{e-1+\sec x} \cdot (1+\sin x) + \ln(e-1+\sec x) \cdot \cos x \cdot \frac{2\sqrt{1+\sin x}}{\cos x}$

$0 - 1 + 1 \qquad 0 \cdot 1 + 1 \cdot \cdot 2$

$= 2$

$\sqrt{x} = y$

Q4 (15 p.) A point P is moving along the curve $x = y^2$ in the first quadrant forming a triangle with vertices $O = (0, 0)$, $P_0 = (x, 0)$ and $P = (x, y)$. How fast is its area changing when the speed of the point P_0 is 4/3 m/min and the distance between O and P is $3\sqrt{10}$.



$\dfrac{dP_0}{dt} = \dfrac{4}{3}$

$\left(\dfrac{dx}{dt}\right)$

$\sqrt{x+x^2} = \sqrt{90}$

$x^2 + x - 90 = 0$

$(x - 10)$
$(x + 9)$

$10, -9$

$y = \sqrt{10}$

$\text{Area} = \dfrac{|y.x|}{2}$

speed of change of area $\dfrac{d\, xy/2}{dt} = \dfrac{dxy}{dt} \cdot \dfrac{1}{2}$

$\dfrac{dx}{dt} \cdot y \cdot \dfrac{1}{2}$

$\dfrac{4}{3} \cdot \sqrt{10} \cdot \dfrac{1}{2}$

$\dfrac{2\sqrt{10}}{3}$

02

Q5 (30 p.) Identifying the domain and all asymptotes, investigating all critical points, intervals where the curve is increasing and where it is decreasing, points of inflections and the concavity of the curve, sketch the curve

$$y = e^{x/(x+1)}.$$

resolving to check

$y = e^{x/(x+1)}$

$\ln y = \ln e \cdot \dfrac{x}{x+1}$

$\ln y = \dfrac{x}{x+1}$

$\dfrac{1}{y} = 1 \cdot (x+1)^{-1} + -(x+1)^{-2} \cdot x$

$\dfrac{1}{y} = \dfrac{x+1}{(x+1)^2} - \dfrac{x}{(x+1)^2} = \dfrac{1}{(x+1)^2}$

$y = (x+1)^2$

3

MAIN SOLUTION

$y = e^{x/(x+1)}$

$\ln y = \ln e \cdot x/(x+1)$

$\ln y = \left(\dfrac{x}{x+1}\right)'$

y'/y

$\dfrac{1}{y} = 1 \cdot (x+1)^{-1} + x \cdot -(x+1)^{-2}$

$= \dfrac{(x+1)}{(x+1)^2} - \dfrac{x}{(x+1)^2}$

$\dfrac{1}{y} = \dfrac{1}{(x+1)^2}$

$y = (x+1)^2 \quad -1$

$y' = 2x + 2 \quad -1$

$y'' = 2$

Singular point = no singular point

critical point = $x = -1$

		-1	
$f(x)$	$+$		$+$
$f'(x)$	$-$		$+$
$f''(x)$	$+$		$+$

→ Its upward

increasing = $[-1, \infty)$

decreasing = $(-\infty, -1]$

upward = $\mathbb{R}$

?

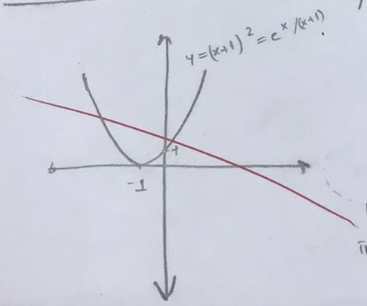
horizontal asymptots: $\lim\limits_{x \to \infty} (x+1)^2 = \infty$ no hor. as.

vertical asymptotes: no sing. point no ver. as.

oblique asymptotes:

$$\begin{array}{l|l} x^2+2x+1 & 2x+2 \\ -\;x^2+x & \frac{x}{2}+\frac{1}{2} \\ \quad x+1 & \\ -\;x+1 & \\ \quad 0 & \end{array}$$

$\dfrac{x}{2} + \dfrac{1}{2}$ oblique asymp.



Inflection point: Its always upward so no inflection point