MATH 124
1st Exam

April, 16; 21.

1) (a) Find the area of the region between the circle $r = 5\cos\theta$ and the curve $r = 2 + \cos\theta$.

(b). Let $g(x, y) = \dfrac{4x^2y^2}{x^2+y^2}$. Find domain of $g$ and sketch it on plane. Also find $\lim\limits_{(x,y)\to(0,0)} g(x, y) = ?$

2) (a) Find the equation of the plane that contains point $P(1, -1, 2)$ and has the normal $n = 2i + 4j + 3k$.

(b) Sketch the plane on space which was obtained in (a).

3). (a) $u = i - 2j + ak$ $(a \in \mathbb{R})$, $V = 2i + aj - 5k$, $w = i - j + 2k$ are given. Determine values of $a$ such that $u \cdot (w \times V) = 4$.

(b). Compute $\int F(t)\,dt$ where $F(t) = t\sqrt{1+t^2}\,i + (te^{-t})j + (\cos^2 t)k$.

4) (a) Let $W = \ln(x^2 + y^2)$; $x = e^{-v}\cos u$, $y = e^{v}\sin u$. Evaluate $\dfrac{\partial W}{\partial u}$, $\dfrac{\partial W}{\partial v}$ at $(u, v) = (\pi/6, 0)$.

(b). Find the directional derivative of $f(x, y) = \dfrac{e^{-x}}{x^2 y}$ at $P(-2, 1)$ in the direction $u = i - j$.

5) (a) Find the equation of the tangent plane to the surface $x^3 e^{y^3 + 2z} + 1 = 0$ at $(-1, 0, 0)$.

(b). Find extremum values of the following function;

$$f(x, y) = xy(4 - x - y).$$

MATH 124 July, 22, 19

1st Exam.

1) Find the area of the region between the circle $r = 5\cos\theta$ and the curve $r = 2 + \cos\theta$ (so called "limaçon")

2) (a) Find the equation of the plane that contains point $P = (1, -1, 2)$ and has the normal $n = 2i + 4j + 3k$.

(b) Sketch the plane on space which was obtained in (a).

3) (a) $u = i - 2j + ak$ $(a \in \mathbb{R})$, $v = 2i + aj - 5k$, $w = i - j + 2k$ are given. Determine values of $a$ such that

$$u \cdot (v \times w) = 4.$$

(b) Let $u = bi + j + k$ and $v = i - j + 2k$ $(b \in \mathbb{R})$ be two vectors in space. Find the value of $b$ such that the angle between $u$ and $v$ is $\pi/3$.

4) (a) Find $\lim_{t \to 0} F(t)$ where $F(t) = \frac{\sin t}{2t} i - \frac{(1 - \cos t)}{t} j + \frac{(t+1)}{e^t} k$.

(b) Compute $\int F(t)\,dt$ if $F(t) = t\sqrt{1+t^2}\, i + (te^{-t}) j + (\sin^2 t) k$.

5) (a) Let $g(x, y) = \frac{4x^2y^2}{x^2 + y^2}$. Find the domain of $g$, and sketch it on the plane.

(b) $$f(x,y) = \begin{cases} g(x,y) & \text{if } (x,y) \neq (0,0) \\ A & \text{if } (x,y) = (0,0). \end{cases}$$

is defined. Determine the value of $A$ such that $f(x,y)$ is continuous at $(0,0)$. Justify your steps.

Good Luck

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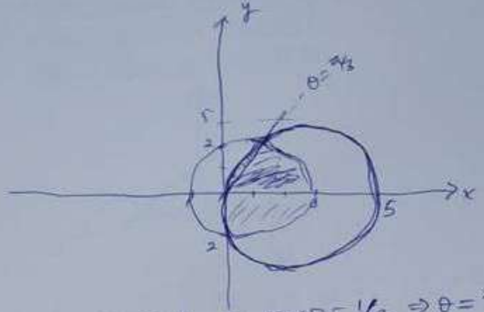
## Solutions of 1st Exam.



1)

| $\theta$ | 0 | $\pi/2$ | $\pi$ | $3\pi/2$ | $2\pi$ |
|---|---|---|---|---|---|
| $r=5\cos\theta$ | 5 | 0 | $-5$ | 0 | 5 |
| $r=2+\cos\theta$ | 3 | 2 | 1 | 2 | 3 |




$5\cos\theta = 2+\cos\theta \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pi/3.$

$$A = 2\left(\frac{1}{2}\int_0^{\pi/3}(2+\cos\theta)^2 d\theta + \frac{1}{2}\int_{\pi/3}^{\pi/2}(5\cos\theta)^2 d\theta\right)$$

$$= \int_0^{\pi/3}(4+4\cos\theta+\cos^2\theta)\,d\theta + \int_{\pi/3}^{\pi/2} 25\cos^2\theta\, d\theta$$

$$= \left[4\theta + 4\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_0^{\pi/3}\right] + 25\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_{\pi/3}^{\pi/2}\right]$$

$$= \left[\frac{4\pi}{3} + \frac{4\sqrt{3}}{2} + \frac{\pi}{6} + \frac{\sqrt{3}}{8} - 0\right] + 25\left[\frac{\pi}{4} + 0 - \frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$$

$$= \frac{43\pi}{12} - \sqrt{3}$$

2) (a) Let $Q=(x,y,z)$ be any point in the plane then

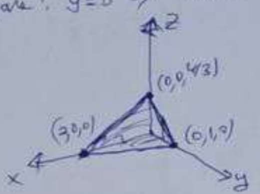
$\vec{PQ}\cdot n=0 \Rightarrow (x-1)2+(y+1)4+(z-2)3=0$ i.e.,

$2x+4y+3z=4.$

(b) $xy$-plane : $z=0 \Rightarrow 2x+4y=4$

$yz$-plane : $x=0 \Rightarrow 4y+3z=4$

$xz$-plane : $y=0 \Rightarrow 2x+3z=0$




3) (a) $u\cdot(v\times w)=\det\begin{bmatrix}1 & -2 & a\\ 2 & a & -5\\ 1 & -1 & 2\end{bmatrix}=\begin{vmatrix}1 & -2 & a\\ 2 & a & -5\\ 1 & -1 & 2\end{vmatrix}\begin{matrix}1 & -2\\ 2 & a\\ 1 & -1\end{matrix}=(2a+10$

$-2a)-(a^2+5-8)=10-a^2+3=4 \Rightarrow a^2=9 \Rightarrow \underline{a=\pm 3}.$

(b) $u\cdot v=\|u\|.\|v\|.\cos\theta$

$b-1+2=\sqrt{b^2+2}.\sqrt{6}.\frac{1}{2} \Rightarrow b+1=\frac{\sqrt{6b^2+12}}{2}$

$\Rightarrow \sqrt{6b^2+12}=2b+2 \Rightarrow 6b^2+12=4b^2+8b+4$

$\Rightarrow 2b^2-8b+8=0$

$\Rightarrow b^2-4b+4=(b-2)^2=0 \Rightarrow \underline{b=2}$

4) (a) $\lim_{t\to 0} F(t)=\left(\lim_{t\to 0}\frac{\sin t}{2t}\right)i-\left(\lim_{t\to 0}\frac{1-\cos t}{t}\right)j+\left(\lim_{t\to 0}\frac{t+1}{e^t}\right)k$

$=\lim_{t\to 0}\frac{1}{2}\left(\frac{\sin t}{t}\right)i-\lim_{t\to 0}(\sin t)j+\lim_{t\to 0}\left(\frac{t+1}{e^t}\right)k$

$=\frac{1}{2}i+0j+k=\frac{i}{2}+k.$

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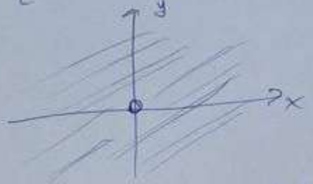
(d) $\int F(t)\,dt = \left(\int t\sqrt{1+t^2}\,dt\right)i + \left(\int te^t\,dt\right)j +$
$+ \left(\int \sin^2 t\,dt\right)k = \frac{1}{2}\frac{2}{3}(1+t^2)^{3/2} i + \left[te^t - \int e^t\,dt\right]j +$
$+ \left(\int \frac{1-\cos 2t}{2}\,dt\right)k = \frac{1}{3}(1+t^2)^{3/2} i + \left[te^t - e^t\right]j + \left(\frac{1}{2}t - \frac{1}{4}\right.$
$\left.\sin 2t\right]k + c$

5) (a) $D_g = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \neq 0\} = \mathbb{R}^2 - \{(0,0)\}$




(b) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = A$ must hold. Now

$\lim\limits_{\substack{x\to 0\\ y=0}} f(x,y) = 0 = \lim\limits_{\substack{y\to 0\\ x=0}} f(x,y)$ is clear.

let $y = mx \Rightarrow \lim\limits_{x\to 0} f(x,mx) = \lim\limits_{x\to 0} \frac{4x^2(m^2x^2)}{x^2+m^2x^2} = 0$.

let $y = mx^2 \Rightarrow \lim\limits_{x\to 0} f(x,mx^2) = \lim\limits_{x\to 0} \frac{4x^2(mx^4)}{x^2+m^2x^4} = 0$. So.

Given $\varepsilon > 0$, we want to find $\delta > 0$ st.

$0 < \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - 0| < \varepsilon$ ie

$|f(x,y) - 0| = \left|\frac{4x^2y^2}{x^2+y^2}\right| = \frac{4(x^2y^2)}{x^2+y^2} \leq \frac{4\cancel{(x^2+y^2)}(x^2+y^2)}{\cancel{(x^2+y^2)}} < 4\delta^2$.

Now $\varepsilon = 4\delta^2 \Rightarrow \delta^2 = \frac{\varepsilon}{4} \Rightarrow \delta = \sqrt{\varepsilon}/2$ can be taken. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = A$.

MATH 124

Second Exam.

1) Let $W = \ln(x^2+y^2)$; $x = e^{-v}\cos u$, $y = e^{v}\sin u$.
Evaluate $\frac{\partial w}{\partial u}$, $\frac{\partial w}{\partial v}$ at $(u,v) = (\frac{\pi}{6}, 0)$.

2) Find extremum values of $f(x,y) = xy(4-x-y)$.

3) $f(x,y,z) = x^3 e^{y^3+2z}$ is given.

(a) Compute $\nabla f(1,1,1)$.

(b) Find the equation of the tangent plane to the surface $x^3 e^{y^3+2z} + 1 = 0$ at $(-1,0,0)$.

4) a) Find the directional derivative of $f(x,y) = \frac{e^{-x}}{x^2 y}$ at $P = (-2,1)$ in the direction $u = -i + j$.

(b) $f(x,y) = x^2 - 2y + y^2$ is given. Find the maximum value of $f$ subject to $x^2+y^2=4$.

5) (a) Compute $\int_{-1}^{2}\int_{x^2-2}^{x} x^2 y \, dy\, dx$

(b). Evaluate $\iint_D (x^2+y^2)\, dA$; where $D$ is bounded by the $x$-axis, the line $y=x$ and the circle $x^2+y^2=1$.

Good Luck

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MATH 124 Second Exam Solutions.

1) $u=\pi/6,\ v=0 \Rightarrow x=\cos\pi/6=\sqrt{3}/2,\ \sin\pi/6=y=1/2$. Now

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}\cdot\frac{dx}{du}+\frac{\partial w}{\partial y}\cdot\frac{dy}{du}=\frac{2x}{x^2+y^2}(-e^{-v}\sin u)+\frac{2y}{x^2+y^2}(e^{v}\cos u)=$$

$$=\frac{-2(e^{-v}\cos u)(e^{-v}\sin u)+2e^{v}\sin u\, e^{v}\cos u}{e^{-2v}\cos^2u+e^{2v}\sin^2u}=\frac{2\sin u\cos u(e^{2v}-e^{-2v})}{e^{-2v}\cos^2u+e^{2v}\sin^2u}$$

$$=\frac{2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\cancel{(e^0-e^0)}^{\,0}}{1}=0.$$ Similarly:

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{-2e^{-2v}\cos^2u+2e^{2v}\sin^2u}{e^{-2v}\cos^2u+e^{2v}\sin^2u}=\frac{-2.\frac{3}{4}+2.\frac{1}{4}}{1}=-1.$$

2) $f_x=y(4-x-y)+(xy)(-1)=y(4-2x-y)$

$f_y=x(4-x-y)+xy(-1)=x(4-x-2y)$.

$f_x=0,\ f_y=0 \Rightarrow \begin{cases} y=0 \text{ or} \\ y=4-2x \end{cases},\ \begin{cases} x=0 \\ x=4-2y \end{cases}$.

Hence $(0,0),(4,0),(0,4),(4/3,4/3)$ are critical points.

$A=f_{xx}=-2y,\ C=f_{yy}=-2x,\ B=f_{xy}=4-2x-2y$. Thus

$D=AC-B^2=4xy-(4-2x-2y)^2$. So

At $(0,0),(4,0),(0,4)$: $D=-16<0 \Rightarrow (0,0),(4,0),(0,4)$ are saddle points of $f$.

At $(4/3,4/3)$: $D=16/3>0$ and $A=-8/3 \Rightarrow (4/3,4/3)$ is a local max. point and the maximum value is $f(4/3,4/3)=64/27$.

3) (a). $\nabla f=(3x^2e^{y^3+2z})i+(3x^3y^2e^{y^3+2z})j+(2x^3e^{y^3+2z})k$

So $\nabla f(1,1,1)=3e^3i+3e^3j+2e^3k$

(b). Let $F = x^3 e^{y^3+2z} + 1$. Then
$F_x = 3x^2 e^{y^3+2z}$, $F_y = 3x^3y^2e^{y^3+2z}$, $F_z = 2x^3e^{y^3+2z}$. So
$F_x(-1,0,0) = 3$, $F_y(-1,0,0) = 0$, $F_z(-1,0,0) = -2$. Hence
$3(x+1) + 0(y-0) + (-2)(z-0) = 0$ i.e. $3x - 2z + 3 = 0$ is the equation of the tangent plane.

4) (a) $u = -i + j \Rightarrow v = \frac{-1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$ is a unit vector. Now
$$\nabla f = \frac{-e^{-x}x^2y - 2xye^{-x}}{x^4y^2} i + \left(\frac{e^{-x}}{x^2}\right)\left(-\frac{1}{y^2}\right) j.$$ So
$$\nabla f(-2,1) = \frac{-e^2 4 + 4e^2}{16} i + \frac{e^2}{4}(-1) j = -\frac{e^2}{4} j.$$
Here $D_v f(-2,1) = \nabla f(-2,1) \cdot v = 0 + \frac{1}{\sqrt{2}}\left(-\frac{e^2}{4}\right) = \frac{-e^2}{4\sqrt{2}}$.

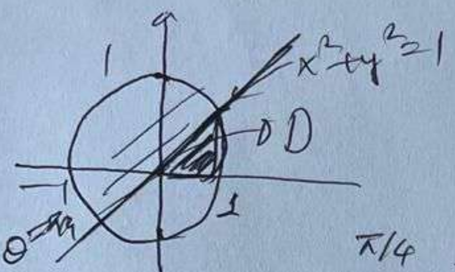
(b) $f = x^2 - 2y - y^2$ (Not $x^2 - 2y + y^2$ ——— ). So
$f_x = 2x$, $f_y = -2 - 2y$, $g_x = 2x$, $g_y = 2y$. Here
$$\begin{cases} 2x = \lambda(2x) \\ -2-2y = \lambda(2y) \\ x^2+y^2-4 = 0 \end{cases} \Rightarrow \lambda = 1 \Rightarrow -2 - 2y = 2y \Rightarrow \boxed{y = -1/2}. \Rightarrow x = \pm \sqrt{15}/2$$
So $(\sqrt{15}/2, -1/2)$, $(-\sqrt{15}/2, -1/2)$. Here the maximum value is $f(\pm\frac{\sqrt{15}}{2}, -1/2) = \frac{15}{4} + 1 + \frac{1}{4} = 5$.

5) (a). $$\int_{-1}^{2}\int_{x^2-2}^{x} x^2y \, dy\,dx = \int_{-1}^{2}\left(x^2 \frac{y^2}{2}\Big|_{x^2-2}^{x}\right)dx = \int_{-1}^{2}\frac{x^2}{2}\left[x^2 - (x^2-2)^2\right]dx = \frac{1}{2}\int_{-1}^{2} x^2(x^2 - x^4 + 4x^2 - 4)\,dx =$$

$$= \frac{1}{2}\int_{-1}^{2}(5x^4 - x^6 - 4x^2)\,dx = \frac{1}{2}\left[x^5 - \frac{x^7}{7} - \frac{4x^3}{3}\Big|_{-1}^{2}\right]$$

$$= \frac{1}{2}\left[2^5 - \frac{2^7}{7} - \frac{4\cdot 2^3}{3}\right] - \frac{1}{2}\left[-1 + \frac{1}{7} + \frac{4}{3}\right].$$

(b).




$$\iint_D (x^2+y^2)\,dA = \int_0^{\pi/4}\int_0^1 r^2 r\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\Big|_0^1\right]d\theta$$

$$= \int_0^{\pi/4}\left(\frac{1}{4} - 0\right)d\theta = \frac{\theta}{4}\Big|_0^{\pi/4} = \frac{\pi}{16}.$$

MATH 124

Final

13, June; 25.

1) a) $f(x,y)=\dfrac{3xy}{y-2|x|}$ is given. Find domain of $f$ and sketch it. Also, find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

b) Sketch the plane that contains point $P(1,2,-1)$ and has the normal $n = 2i - j + k$.

2) a) $u = i+2j-ak$, $v = 2i+aj+3k$, $w = i-j+k$ $(a\in\mathbb{R})$ are given. Find the values of $a$ such that $v\cdot(w\times u)=0$.

b) Let $F(t) = (t\sqrt{t})\,i + te^{t}\,j + (\sin^2 2t)\,k$. Find $\int F(t)\,dt$.

3) a) Let $W = x^3y^2 + \ln(x+y)$ and $x = e^{v}\ln u$, $y = e^{-v}\ln u$. Evaluate $\dfrac{\partial W}{\partial u}$ and $\dfrac{\partial W}{\partial v}$ at $(u,v)=(e,0)$.

b) Let $f(x,y)=\sqrt{x^2+y^2} - e^{xy}$ and $u = 3i+4j$. Find directional derivative of $f$ at $P(1,1)$ in the direction of $u$.

4) a) Find the minimum value of $f(x,y,z)=x^2+y^2+z^2$ subject to $x-2y+3z=4$.

b) Find the volume of the solid which is under the plane $z = x+2y+4$ and above the region $D$ bounded by the lines $y=2x$, $y=3-x$ and $y=0$.

5) a) Compute $\iint\limits_D (x^2+y^2)\,dA$ where $D$ is bounded by the $x$-axis, the line $y=x$ and the circle $x^2+y^2=1$.

b) Evaluate $\iint\limits_D y\,dx\,dy$ where $D$ is the region whose boundary $C$ is the parallelogram with sides along the lines $x+3y=-1$, $x+3y=3$, $x-y=1$ and $x-y=2$.

Good Luck.

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MATH 124 Aug., 19, 19
Final

1) (a) $\alpha = i - 2j + tk \ (t \in \mathbb{R})$, $\beta = 2i + tj - 5k$ and $\gamma = i - j + 2k$ are given. Find the values of $t$ such that $\alpha \cdot (\beta \times \gamma) = 4$.

(b) Find the equation of the plane that contains point $P = (1, -1, 2)$ and has the normal $n = 2i + 4j + 3k$.

2) Let $f(x,y) = \dfrac{9x^2y^2}{x^2+y^2}$. Then.

(a) Find domain of $f$ and sketch it on the plane.

(b) Find $\lim\limits_{P \to (0,0)} f(x,y)$.

3) (a) Find the directional derivative of $f(x,y) = e^{-x} - x^2y$ at $P = (-2, 1)$ in the direction $u = -i + j$.

(b). $g(x,y) = x^2 - 2y - y^2$ is given. Find the maximum value of $g$ subject to $x^2 + y^2 = 4$.

4). Let $T = \ln(x^2 + y^2)$; $x = e^{v}\cos u$, $y = e^{-v}\sin u$.
Evaluate $\dfrac{\partial T}{\partial u}$, $\dfrac{\partial T}{\partial v}$ at $(u, v) = (\pi/3, 0)$.

5). Evaluate $\iint\limits_D x^2 y^{-4} \, dx\,dy$, where $D$ is the region bounded by the hyperbolas: $xy = 2$, $xy = 4$ and the parabolas: $y^2 = x$, $y^2 = 3x$.

Good Luck

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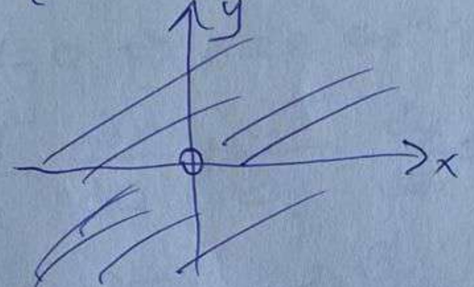
1) (a) $\alpha\cdot(\beta\times\gamma) = \det\begin{bmatrix} 1 & -2 & t \\ 2 & t & -5 \\ 1 & -1 & 2 \end{bmatrix} = 10 - t^2 + 3 = 4$
$\Rightarrow \boxed{t = \pm 3}$

(b). $(x-1)2 + (y+1)4 + (z-2)3 = 0$ so,
$2x + 4y + 3z = 4$.

2).(a) $D_f = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\neq 0\} = \mathbb{R}^2 - \{(0,0)\}$




(b). $\lim\limits_{x\to 0}$ ---- $= 0$. So
Given $\varepsilon>0$; $0<\sqrt{x^2+y^2}<\delta \Rightarrow |f(x,y)-0|<\varepsilon$ so,

$$|f(x,y)| = \frac{9x^2y^2}{x^2+y^2} \le \frac{9(x^2+y^2)\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} < 9\delta^2$$

Now $\varepsilon = 9\delta^2 \Rightarrow \delta = \sqrt{\varepsilon}/3$.

3) a) $u = -i + j \Rightarrow v = \frac{u}{\|u\|} = \frac{-1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$.

$f_x = -e^{-x} - 2xy$, $f_y = -x^2$. So
$D_v f(-2,1) = (-e^2+4)\cdot\left(\frac{-1}{\sqrt{2}}\right) + (-4)\left(\frac{1}{\sqrt{2}}\right) = \frac{e^2}{\sqrt{2}} - \frac{4}{\sqrt{2}} - \frac{4}{\sqrt{2}}$

$= \frac{e^2-8}{\sqrt{2}}$.

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b) $f = x^2 - 2y - y^2$, $g(x,y) = x^2 + y^2 - 4$

$f_x = 2x$, $f_y = -2 - 2y$.

$$\begin{cases} f_x = \lambda g_x, \ f_y = \lambda g_y \\ x^2 + y^2 - 4 = 0 \end{cases} \Rightarrow \begin{cases} 2x = \lambda 2x \\ -2 - 2y = \lambda 2y \\ x^2 + y^2 - 4 = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 1 \\ y = -1/2 \\ x^2 + \frac{1}{4} - 4 = 0 \end{cases}$$

$\Rightarrow x = \sqrt{15}/2, -\sqrt{15}/2$.

$f(\pm\sqrt{15}/2, -1/2) = 9/2$.

4) $u = \pi/3$, $v = 0 \Rightarrow x = e^0 \cos \pi/3 = 1/2$, $y = e^0 \sin \pi/3 = \sqrt{3}/2$.

$$\frac{\partial T}{\partial u} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{2x}{x^2+y^2}(-e^v \sin u) + \frac{2y}{x^2+y^2} e^v \cos u$$

$$= \frac{2 \cdot 1/2}{\frac{1}{4} + \frac{3}{4}} \cdot \left(-e^0 \cdot \frac{\sqrt{3}}{2}\right) + \frac{2 \cdot \sqrt{3}/2}{\frac{1}{4} + \frac{3}{4}} \cdot \left(e^0 \cdot 1/2\right)$$

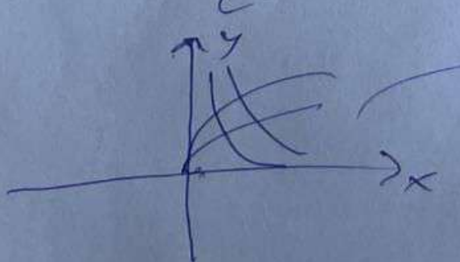
$$= \frac{-\sqrt{3}/2}{1} + \frac{\sqrt{3}/2}{1} = 0.$$

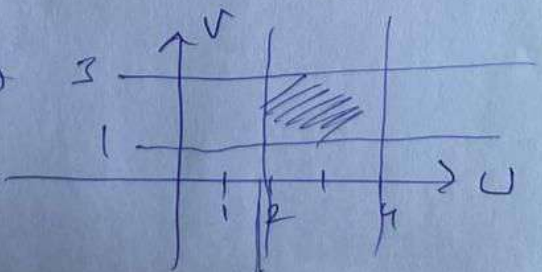
$$\frac{\partial T}{\partial v} = \frac{\partial T}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{2x}{x^2+y^2} \cdot (e^v \cos u) +$$

$$+ \frac{2y}{x^2+y^2}(-e^v \sin u) = 1 \cdot \left(e^0 \cdot \frac{1}{2}\right) + 1 \cdot \left(-e^0 \cdot \frac{\sqrt{3}}{2}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2}$$

$$= \frac{1 - \sqrt{3}}{2}.$$

5). Let $\begin{cases} xy = 2 \\ xy = 4 \end{cases}$, $\begin{cases} y^2 = x \\ y^2 = 3x \end{cases} \Rightarrow$ let $\begin{cases} xy = u \\ \frac{y^2}{x} = v. \end{cases}$ Then



$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} y & x \\ -y^2/x^2 & 2y/x \end{bmatrix} = \frac{3y^2}{x}$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3y^2/x} = \frac{x}{3y^2} = \frac{1}{3v}.$$ So,

$$\iint_R x^2 y^{-4}\, dx\,dy = \int_2^4 \int_1^3 \frac{1}{3v}\cdot\frac{1}{v^2}\, dv\, du$$

$$= \frac{1}{3}\int_2^4 \int_1^3 \frac{dv}{v^3}\, du = \frac{1}{3}\int_2^4 \left( -\frac{1}{2v^2}\Big|_1^3 \right) du$$

$$= -\frac{1}{6}\int_2^4 \left( \frac{1}{v^2}\Big|_1^3 \right) du = -\frac{1}{6}\int_2^4 \left( \frac{1}{9} - 1 \right) du$$

$$= -\frac{2}{6}\left( \frac{1}{9} - 1 \right) = 8/27.$$

MATH 124
Midterm

April, 1; 26

1) Find the volume of a solid which is obtained by revolving the region $y = \ln x$, $1 \le x \le \sqrt{e}$, about the $y$-axis:
(a) by Shell's method, (b) by Washer's method.

2) a) Find the arc length of the curve $y = x^4 + \frac{1}{32x^2}$ from $x = 1$ to $x = 3$.

b) Find the area of surface generated when the arc of the curve $y = \frac{x^3}{3} + \frac{1}{4x}$ between $x = 1$ and $x = 2$ is revolved about the $x$-axis?

3) a) Determine convergence of the sequence $a_n = \frac{n}{3^n}$

b) Determine convergence of the series $\sum_{n=2} 2\left(\frac{2}{3}\right)^n$.

4) a) $\sum_1^{\infty} \frac{k}{2^k} x^k$ is given, Find the radius of convergence and all points where the series converges?

b) Find the area of the region that is inside the circle $r = \sin\theta$ and outside the cardioid $r = 1 - \cos\theta$.

5) a) For what values of $t$ is the angle between the vectors $u = ti - j$ and $v = i + j$ equal to $\pi/3$.

b) Let $u = i - 2j + 6k$, $v = 3i + j + 5k$, $w = 4i + j - 2k$.
Find $u \cdot (v \times w)$.

Good Luck

| 1 | 2 | 3 | 4 | 5 | |
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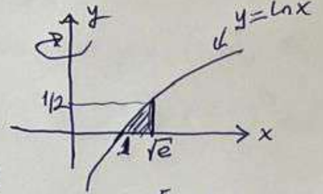
SOLUTIONS 

1)




a) $V = 2\pi \int_1^{\sqrt{e}} x \ln x \, dx = 2\pi \left[ \frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_1^{\sqrt{e}} = 2\pi \left[ \frac{e}{4} - \frac{e}{4} + \frac{1}{4} \right] = \pi/2$

b) $V = \pi \int_0^{1/2} \left( (\sqrt{e})^2 - (e^y)^2 \right) dy = \pi \int_0^{1/2} (e - e^{2y}) \, dy = ey - \frac{e^{2y}}{2} \Big|_0^{1/2} \pi$

$= \pi/2.$

2) a) $y = x^4 + \frac{1}{32x^2} \Rightarrow y' = 4x^3 - \frac{1}{16x^3}$. Thus

$S = \int_1^3 \sqrt{1 + (y')^2} \, dx = \int_1^3 \sqrt{1 + \left(4x^3 - \frac{1}{16x^3}\right)^2} \, dx = \int_1^3 \sqrt{\left(4x^3 + \frac{1}{16x^3}\right)^2} \, dx$

$= \int_1^3 \left(4x^3 + \frac{1}{16x^3}\right) dx = \left(x^4 - \frac{1}{32x^2}\right) \Big|_1^3 = \left(81 - \frac{1}{288} - 1 + \frac{1}{32}\right)$

$= 2252/288 = 563/72.$

b) $S = 2\pi \int_1^2 x \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2} \, dx = 2\pi \int_1^2 x \left(x^2 + \frac{1}{4x^2}\right) dx$

$= 2\pi \int_1^2 \left(x^3 + \frac{1}{4x}\right) dx = 2\pi \left[ \frac{x^4}{4} + \frac{1}{4} \ln x \Big|_1^2 \right] =$

$= 2\pi \left[ 4 + \frac{1}{4} \ln 2 - \frac{1}{4} - 0 \right] = 2\pi \left[ \frac{16 + \ln 2 - 1}{4} \right]$

$= \frac{2\pi}{4} \left[ 15 + \ln 2 \right]$

$= \frac{\pi}{2} (15 + \ln 2)$

2.

3) a) $a_n = \frac{n}{3^n} \Rightarrow \lim a_n = \lim \frac{n}{3^n} = \lim \frac{1}{3^n \ln 3} = 0.$

So, $a_n$ converges. (OR, $a_{n+1} < a_n$ and $0 < a_n < \frac{1}{3}$)

b) $\sum_2^\infty 2(2/3)^n = \sum_0^\infty 2(2/3)^n - (2+\frac{4}{3}) = \frac{2}{1-\frac{2}{3}} - \frac{10}{3}$

$= 6 - \frac{10}{3} = 8/3.$

4) a) $\sum_0^\infty \frac{k}{2^k} x^k \Rightarrow R = \lim \left| \frac{k}{2^k} \cdot \frac{2^{k+1}}{k+1} \right| = \lim \frac{k}{k+1} \cdot 2 = 2$

So, the series converges on $(-2,2)$ ie, $|x| < 2$.

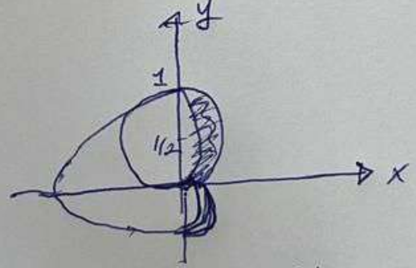
Now, put $x=-2$. Then $\sum \frac{k}{2^k}(-2)^k = \sum k(-1)$ Div.

put $x=2$. Then $\sum \frac{k}{2^k}(2)^k = \sum k$ Div.

Hence, the series converges on $(-2,2)$.

b) $r = \sin\theta \Rightarrow x^2 + (y-1/2)^2 = 1/4$ ie, $C(0,1/2)$, radius $= 1/2$.

| $\theta$ | $0$ | $\pi/2$ | $\pi$ | $3\pi/2$ | $2\pi$ |
|---|---|---|---|---|---|
| $r=1-\cos\theta$ | $0$ | $1$ | $2$ | $1$ | $0$ |




$A = \frac{1}{2}\int_0^{\pi/2} (\sin^2\theta - (1-\cos\theta)^2)\, d\theta = \frac{1}{2}\int_0^{\pi/2} (2\sin^2\theta - 2 + 2\cos\theta)\, d\theta$

$= \int_0^{\pi/2} (\sin^2\theta - 1 + \cos\theta)\, d\theta = \frac{1}{2}(\theta - \frac{\sin 2\theta}{2}) - \theta + \sin\theta \Big|_0^{\pi/2}$

$= \frac{1}{2}(\frac{\pi}{2} - 0) - \frac{\pi}{2} + 1 - 0 = 1 - \frac{\pi}{4}$.

$A = \frac{1}{2}\int_0 \ldots \cos\theta)$

$= \int_0^{\pi/2} (\sin^2\theta - 1 + \cos\theta) \ldots - 0 = 1 - \frac{\pi}{4}$

$= \frac{1}{2}\left(\frac{\pi}{2} - 0\right) - \frac{\pi}{2} + 1 - 0$



5) a) $u \cdot v = \|u\| \|v\| \cos\theta$

$\Rightarrow t - 1 = \sqrt{t^2+1}\,\sqrt{2} \cdot \frac{1}{2}$

$\Rightarrow 2t - 2 = \sqrt{2t^2+2} \Rightarrow 4t^2 - 8t + 4 = 2t^2 + 2$

$\Rightarrow 2t^2 - 8t + 2 = 0 \Rightarrow t^2 - 4t + 1 = 0$

$\Rightarrow t = \frac{4 \mp \sqrt{12}}{2} = 2 \mp \sqrt{3}.$

b) $v \times w = \begin{vmatrix} i & j & k \\ 3 & 1 & 5 \\ 4 & 1 & -2 \end{vmatrix} = i(-2-5) - j(-6-20) + k(3-4)$

$= -7i + 26j - k.$

Thus $u \cdot (v \times w) = 1.(-7) - 2(+26) + 6(-1) = -7 - 52 - 6$

$= -65$

MATH 124.

Midterm Exam

Apr. 30, 25

1) a) $f(x,y)=\dfrac{xy}{y+3|x|}$ is given. Find domain of $f$ and sketch it, also find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

b) Find the area of the region between the curves $r=2\cos\theta$ and $r=1+\cos\theta$.

2) a) Sketch the plane that contains point $P(2,-1,1)$ and has the normal $n=i+j-2k$.

b) $u=i-2j+ak$, $v=2i+aj-3k$, $w=i-2j-k$ $(a\in\mathbb{R})$ are given. Find the values of $a$ such that $u\cdot(v\times w)=0$.

3) a) Let $F(t)=(\cos^2 t)i+e^{-3t}j+(\sec t)k$. Find $\int F(t)\,dt$.

b) $W=xy^2-\ln(x^2y)$ and $x=e^{v}\ln u$, $y=e^{-v}\ln u$ are given. Evaluate $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ at $(u,v)=(e,0)$.

4) a) Let $f(x,y)=x^2+y^2-e^{2xy}$ and $u=2i+j$. Find the directional derivative of $f$ at $P(2,1)$ in the direction of $u$.

b) Find the equation of the tangent plane to the surface $xy^2+\sqrt{xz+2yz}-1=0$ at the point $(1,1,0)$.

5) a) Find absolute extremum values of the function $f(x,y)=xy-x^3y^2$ on the region $R=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1\}$.

b) Find local extremum values of the function $f(x,y)=2xy(2-x-y)$.

Good Luck.

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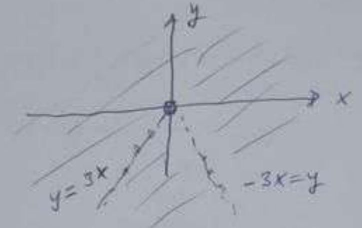
Solutions

1) a) $D = \{(x,y) \mid y \neq -3|x|\} = \{(x,y) \mid \begin{array}{l} y \neq -3x \ (x>0) \\ y \neq 3x \ (x<0) \end{array}\}$




$\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$ $\quad \lim\limits_{x\to 0} f(x,0) = \lim\limits_{y\to 0} f(0,y) = \lim\limits_{x\to 0} f(x,x) =$

$= \lim\limits_{x\to 0} f(x,x^2) = 0.$ Let us show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0.$

Suppose $\varepsilon > 0$, then

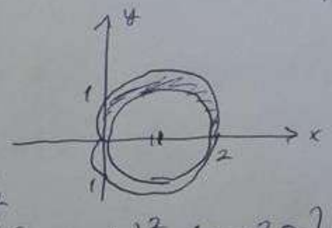
$0 < \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - 0| < \varepsilon$ ?

So $\left| \dfrac{xy}{y+3|x|} \right| = \dfrac{|x||y|}{|y+3|x||} \leq \dfrac{|x||y|}{|y|} = |x| < \sqrt{x^2+y^2} < \delta$

ie we can choose $\delta = \varepsilon$. Hence $\lim f = 0$.

b) $r = 2\cos\theta$, $r = 1+\cos\theta$

| $\theta$ | 0 | $\pi/2$ | $\pi$ | $3\pi/2$ | $2\pi$ |
|---|---|---|---|---|---|
| $r$ | 2 | 1 | 0 | 1 | 2 |




$$A = 2 \cdot \frac{1}{2} \int_0^{\pi/2} \left[ (1+\cos\theta)^2 - 4\cos^2\theta \right] d\theta$$

$$= \int_0^{\pi/2} (1 + 2\cos\theta - 3\cos^2\theta)\, d\theta = \left[ \theta + 2\sin\theta - \frac{3}{2}\theta - \frac{3}{4}\sin 2\theta \right|_0^{\pi/2}$$

$$= \left[ -\frac{\theta}{2} + 2\sin\theta - \frac{3}{4}\sin 2\theta \Big|_0^{\pi/2} \right] = -\frac{\pi}{4} + 2 = \frac{8-\pi}{4}.$$

2) a) $(x-2)+(y+1)+(z-1)(-2)=0$ i.e,

$$x+y-2z=-1.$$


b) $$u \cdot (v \times w) = \begin{vmatrix} 1 & -2 & a \\ 2 & a & -3 \\ 1 & -2 & -1 \end{vmatrix} = 0 \quad \text{ie,}$$

$$(-a-6)+2(-2+3)+a(-4-a)=0$$

$$\Rightarrow -a-6-4+6-4a-a^2 = a^2+5a+4=0$$

$$\Rightarrow a = \frac{-5 \mp \sqrt{9}}{2} = -4, -1.$$

3) a) $$\int f(t)\,dt = \left(\int \cos^2 t\,dt\right) i + \left(\int e^{-3t}\,dt\right) j + \left(\int \sec t\,dt\right) k = \frac{1}{2}\left(t+\frac{1}{2}\sin 2t\right) i - \frac{1}{3}e^{-3t} j + \ln|\sec t + \tan t|\, k + c.$$

b) $W = xy^2 - \ln(x^2 y)$; $x = e^{v}\ln u$, $y = e^{-v}\ln u$

$(u,v)=(e,0) \Rightarrow u=e, v=0 \Rightarrow x=y=1.$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} = \left(y^2 - \frac{2}{x}\right)\frac{e^v}{u} +$$

$$+\left(2xy-\frac{1}{y}\right)\frac{e^{-v}}{u}=(1-2)\frac{1}{e}+(2-1)\frac{1}{e}=0.$$

$$\frac{\partial w}{\partial v}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}=\left(y^2-\frac{2}{x}\right)e^{v}\ln u+$$

$$+\left(2xy-\frac{1}{y}\right)\left(-e^{-v}\ln u\right)=(1-2)1+(2-1)(-1)=-2.$$

4) a) $v=\frac{2}{\sqrt5}i+\frac{1}{\sqrt5}j$, $f_x=2x-2ye^{2xy}$

$f_y=2y-2xe^{2xy}$

So, $D_vf(2,1)=\left[(4-2e^4)i+(2-4e^4)j\right]\cdot\left(\frac{2}{\sqrt5}i+\frac{1}{\sqrt5}j\right)$

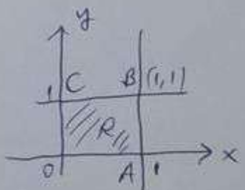
$$=(4-2e^4)\frac{2}{\sqrt5}+(2-4e^4)\frac{1}{\sqrt5}=\frac{8}{\sqrt5}-\frac{4e^4}{\sqrt5}+\frac{2}{\sqrt5}-\frac{4e^4}{\sqrt5}$$

$$=\frac{10}{\sqrt5}-\frac{8e^4}{\sqrt5}.$$

b). $F(x,y,z)=xy^2+\sqrt{xz+2yz}-1$

and $F_x=y^2+\frac{z+0}{2\sqrt{xz+2yz}}\Rightarrow F_x(1,1,0)$ fails to exist.

5) a) $f(x,y)=xy-x^3y^2$




$f_x=y-3x^2y^2$, $f_y=x-2yx^3\Rightarrow f_x=f_y\Rightarrow x=y=0$

ie $(0,0)$ is a CP.

On OA and OC : $f(x,y)=0$ ie there is no CP.

On AB : $x = 1 \Rightarrow f(y) = y - y^2 \Rightarrow f' = 1 - 2y = 0$

$\Rightarrow y = 1/2$

ie $(1, 1/2)$ is a CP.

On CB : $y = 1 \Rightarrow f(x) = x - x^3 \Rightarrow f' = 1 - 3x^2 = 0$

$\Rightarrow x = \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$. Thus

$\left(\frac{1}{\sqrt{3}}, 1\right)$ is a CP but $\left(-1/\sqrt{3}, 1\right) \notin R$.

Now: $f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0$.

$f(1, 1/2) = \frac{1}{2} - \frac{1}{4} = 1/4$.

$f\left(\frac{1}{\sqrt{3}}, 1\right) = \frac{2\sqrt{3}}{9}$.

Hence Absol. max $= \frac{2\sqrt{3}}{9} = f\left(\frac{1}{\sqrt{3}}, 1\right)$

" min $= 0 = f(0,0) = f(1,0) = \cdots$

b) $f = 2xy(2 - x - y) \Rightarrow f_x = y(4 - 4x - 2y)$

$f_y = x(4 - 4y - 2x)$.

$f_x = f_y = 0 \Rightarrow (0,0), (0,2), (2,0), \left(\frac{2}{3}, \frac{2}{3}\right)$ are CP's.

At $(0,0), (2,0), (0,2)$ : $D = -16 < 0$ ie these are saddle points

At $(2/3, 2/3)$ : $D = 48/9 > 0$, $A = -8/3 < 0$. ie

$(2/3, 2/3)$ is a local max. point and the value is

$f(2/3, 2/3) = 16/27$.

MATH124
Resit Exam

1) a) $f(x,y) = \dfrac{2xy}{y+3|x|}$ is given. Find domain of $f$ and sketch it. Also find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

b) Sketch the plane that contains point $P(1,-3,2)$ and has the normal $n = 2i+3j-k$.

2) a) $u = 2i - j + ak$, $v = i + aj + 2k$, $w = i + j - k$ $(a \in \mathbb{R})$ are given. Find the values of $a$ such that $u \cdot (v \times w) = -2$.

b) Let $F(t) = (t\cos t)i + \left(\dfrac{\ln t}{t}\right)j + \left(\cos^2\left(\dfrac{t}{2}\right)\right)k$. Find $\int F(t)\,dt$

3) a) Let $w = x^2y^3 - \ln(x+y)$ and $x = e^v \ln u$, $y = e^{-v}\ln u$.
Evaluate $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ at $(u,v) = (e,0)$.

b) Let $f(x,y) = x\sqrt{x^2+y^2} - e^{x^2y}$ and $u = i + j$. Find directional derivative of $f$ at $P(1,2)$ in the direction of $u$.

4) a) Find local extremum values of $f(x,y) = xy(3-x-y)$.

b) Find the volume of the solid which is under the plane $z = x+2y+4$ and above the region $D$ bounded by the lines $y = 2x$, $y = 3-x$ and $y = 0$.

5) a) Evaluate $\iint_D xy\,dx\,dy$, where $D$ is the region in the first quadrant that lies inside the circle $x^2+y^2 = 4x$.

b) Evaluate $\iint_D x^2y\,dx\,dy$ where $D$ is the region enclosed by $xy = 1$, $xy = 7$ and $y = 2x^2$, $y = 4x^2$.

| 1 | 2 | 3 | 4 | 5 |
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Good Luck.