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MATH-118

(Previous Exam Questions)

FINAL

2024,2023,2022,2019
2018,2017

MATH-118 Final -2024-

* PLEASE WRITE YOUR NAME CLEARLY WITH BALL PEN USING CAPITAL LETTERS *		
FULL NAME	STUDENT ID	DURATION: 120 MINUTES 8 QUESTIONS ON 4 PAGES TOTAL: 100 POINTS
SAMPLE SOLUTION		

Q1. (7+7=14 pts) The question has unrelated parts.

a) Determine whether the series $\sum_{n=1}^{\infty} \underbrace{\frac{\sin n}{e^{2n}}}_{a_n}$ is absolutely convergent, conditionally convergent or divergent. Give your explanations.

* a_n is not always positive $\forall n \in [1,\infty)$

$$0 \leqslant |a_n| = \left|\frac{\sin(n)}{e^{2n}}\right| \leqslant \frac{1}{e^{2n}} = \left(\frac{1}{e^2}\right)^n$$

Since $\sum_{n=1}^{\infty}\left(\frac{1}{e^2}\right)^n$ is convergent by geometric series $\left(\left|\frac{1}{e^2}\right| < 1\right)$,

$\sum_{n=1}^{\infty}\left|\frac{\sin(n)}{e^{2n}}\right|$ is also convergent by Comparison Test. Therefore,

$\sum_{n=1}^{\infty}\frac{\sin(n)}{e^{2n}}$ is absolutely convergent.

b) Evaluate the integral $\int_2^{\infty}\frac{dx}{x\ln x}$, if it exists.

$\int_2^{\infty}\frac{dx}{x\ln x}$ is an improper integral of type I.

$$\int_2^{\infty}\frac{dx}{x\ln x} = \lim_{t\to\infty}\int_2^{t}\frac{dx}{x\ln x} \underset{\boxed{\ln x = u,\ \frac{dx}{x}=du}}{=} \lim_{t\to\infty}\int_{\ln 2}^{\ln t}\frac{du}{u} = \lim_{t\to\infty}\ln|u|\Big|_{u=\ln 2}^{u=\ln t}$$

$$= \lim_{t\to\infty}\ln|\ln t| - \ln|\ln 2| = \infty$$

So, $\int_2^{\infty}\frac{dx}{x\ln x}$ diverges to ∞.

P2 **Q2. (8+8=16 pts)** Evaluate the following limits, if they exist.

a) $\displaystyle \underbrace{\lim_{(x,y)\to(1,0)} \frac{(x-1)^2y^3}{(x-1)^2+4y^2}}_{f(x,y)}$

Approach $(1,0)$ along lines : $y=m(x-1)$. Then :

$$\lim_{(x,y)\to(1,0)} f(x,y) \underset{y=m(x-1)}{=} \lim_{x\to 1} \frac{\cancel{(x-1)^5}\cdot m^3}{\cancel{(x-1)^2}\underbrace{[1+4m^2]}_{>0}} = 0 \text{ (candidate)}$$

$$0 \leq \left| \underbrace{\frac{(x-1)^2}{(x-1)^2+4y^2}}_{\leq 1} y^3 - 0 \right| \leq \underbrace{|y^3|}_{0 \text{ as } (x,y)\to(1,0)} \Rightarrow \text{Thus, by Squeeze } \lim_{(x,y)\to(1,0)} \frac{(x-1)^2y^3}{(x-1)^2+4y^2} = 0$$

b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{5x^3y}{x^6+3y^2}$

Along $y=0$: $\displaystyle \lim_{x\to 0} \frac{5x^3\cdot 0}{x^6+3\cdot 0^2} = \lim_{x\to 0} \frac{0}{x^6} = 0$

Along $x=0$: $\displaystyle \lim_{y\to 0} \frac{5\cdot 0^3\cdot y}{0^6+3y^2} = \lim_{y\to 0} \frac{0}{3y^2} = 0$

} So, if limit exists, it must be 0.

But, along $y=x^3$: $\displaystyle \lim_{x\to 0} \frac{5x^3\cdot x^3}{x^6+3x^6} = \lim_{x\to 0} \frac{5x^6}{4x^6} = \frac{5}{4} \neq 0$

Hence, $\displaystyle \lim_{(x,y)\to(0,0)} \frac{5x^3y}{x^6+3y^2}$ D.N.E

Q3. (6+6=12 pts) Let $f(x,y) = \begin{cases} \dfrac{2x^3-y^3}{x^2+3y^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$

a) Find $f_x(0,0)$, if it exists.

$$\lim_{\substack{h\to 0\\(h\neq 0)}} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{2h^3}{h^2}-0}{h} = \lim_{h\to 0} \frac{2h^3}{h^3} = 2$$

So, $f_x(0,0) = 2$ (exists)

b) Find and write $f_x(x,y)$ for all $(x,y)\in \mathrm{Dom}(f)$. (Do NOT simplify your answer.)

$$f_x(x,y) = \begin{cases} \dfrac{6x^2(x^2+3y^2)-2x(2x^3-y^3)}{(x^2+3y^2)^2} & \text{if } (x,y)\neq(0,0) \\ \underbrace{2}_{\text{by part a}} & \text{if } (x,y)=(0,0) \end{cases}$$

$\nabla g = (y^2, 2xy - 2\sin y - e^{z+1}, -ye^{z+1})$ 2

$\nabla g = (0, -1, 0)$

Name: ______________________

Q4. (5+5+6=16 pts) Let $g(x,y,z) = xy^2 + 2\cos y - ye^{z+1}$ and $P_0 = (1,0,-1)$.

a) Find the gradient $\nabla g(x,y,z)$ at the point P_0.

$\nabla g(x,y,z) = (g_x(x,y,z), g_y(x,y,z), g_z(x,y,z))$

$$\left.\begin{aligned} g_x &= y^2 \\ g_y &= 2xy - 2\sin y - e^{z+1} \\ g_z &= -ye^{z+1} \end{aligned}\right\} \nabla g(1,0,-1) = (g_x(1,0,-1), g_y(1,0,-1), g_z(1,0,-1)) = \underline{(0,-1,0)}$$

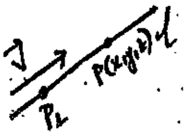
b) Find the directional derivative of $g(x,y,z)$ at point P_0 in the direction of $\vec{u} = \vec{i} + 2\vec{j} - 2\vec{k}$.

g, g_x, g_y, g_z are all continuous $\forall (x,y,z)$, since they are polynomials, trigonometric and exponential functions. So, g is differentiable. Then;

$$D_{\vec{u}}\, g(P_0) = \nabla g(P_0) \cdot \frac{\vec{u}}{\|\vec{u}\|} = (0,-1,0) \cdot \tfrac{1}{3}(1,2,-2) = -\tfrac{2}{3}$$

c) Find the equation of the line passing through the point $P_1 = (-1,2,1)$ and in the direction in which the directional derivative of $g(x,y,z)$ at point P_0 increases most rapidly.

$D_{\vec{u}}\, g(P_0)$ increases most rapidly in the direction of $\nabla g(P_0)$. Thus, the direction vector of the line l $\to$ $\vec{v} = \nabla g(P_0) = (0,-1,0)$ $\to$ by part a.



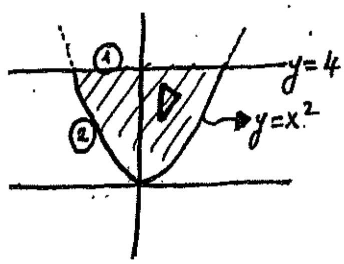
$\vec{P_1P} \parallel \vec{v} \Rightarrow \vec{P_1P} = t\cdot\vec{v} \Rightarrow (x+1, y-2, z-1) = t\cdot(0,-1,0)$

$\Rightarrow \boxed{x=-1,\ y=2-t,\ z=1,\ t\in\mathbb{R}}$

Q5. (15 pts) Find the absolute maximum and absolute minimum values of the function $f(x,y) = 1 + xy - x - y$ on the region bounded by the parabola $y = x^2$ and the line $y = 4$.

f is continuous on a closed and bounded region. Hence abs. max/min exist.

Step 1: Interior D:



$$\text{C.P's: } \left.\begin{aligned} f_x &= y-1=0 \\ f_y &= x-1=0 \end{aligned}\right\} (1,1) \notin \text{Int}(D)$$
So, No C.P's.

Step 2: Boundaries of D:

1 $y=4,\ -2\le x\le 2$: $f(x,4) = 1+4x-x-4 = 3x-3 = g(x) \to$ C.P's of g: $g'(x) = 3 \ne 0$ (No C.P's)

End-points of Dom(g): $\boxed{(-2,4),(2,4)}$

2 $y=x^2,\ -2\le x\le 2$: $f(x,x^2) = 1+x^3-x-x^2 = h(x) \to$ C.P's of h: $h'(x) = 3x^2-2x-1=0$
$\Rightarrow x = -\tfrac{1}{3}, 1 \in [-2,2]$.

End-points of Dom(h): $\boxed{(-2,4),(2,4)}$ $\boxed{(-\tfrac{1}{3},\tfrac{1}{9}),(1,1)}$

Step 3: Comparison: (4-candidates)

$$\left.\begin{aligned} f(-2,4) &= -9 \ (\text{min}) \\ f(2,4) &= 3 \ (\text{max}) \\ f(-\tfrac{1}{3},\tfrac{1}{9}) &= \tfrac{32}{27} \\ f(1,1) &= 0 \end{aligned}\right\}$$

$\boxed{\text{So, abs. max occurs at } f(2,4)=3, \text{ and abs. min occurs at } f(-2,4)=-9}$

Q6. (8 pts) Write the answers to the following questions in the corresponding box. No partial credit will be given to your calculations.

Let D be the region bounded by the curves $y = 5 - x^2$ and $y = (x-1)^2$.

(a) Express the area of the region D as a SINGLE integral (Do NOT evaluate integral.

$$Area(D) = \int_{\boxed{-1}}^{\boxed{2}} \boxed{(5-x^2) - (x-1)^2}\, dx$$

(b) Express the area of the region D as an iterated DOUBLE integral (Do NOT evauate integral.

$$Area(D) = \int_{\boxed{-1}}^{\boxed{2}} \int_{\boxed{(x-1)^2}}^{\boxed{5-x^2}} \boxed{1}\, dydx$$

Q7. (7 pts) Evaluate the double integral $\int_0^8 \int_{y^{1/3}}^{2} e^{x^4}\, dxdy$

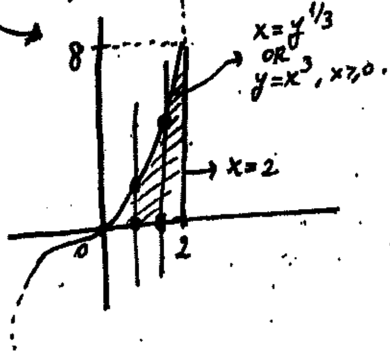
$I = \iint_D e^{x^4}\, dxdy$ where $D = \{(x,y) \mid y^{1/3} \le x \le 2,\ 0 \le y \le 8\}$

If we change the order of integration ($dxdy \to dydx$)

$$I = \int_0^2 \left(\int_0^{x^3} e^{x^4} dy\right) dx = \int_0^2 e^{x^4} \cdot y \Big|_0^{x^3} = \int_0^2 x^3 \cdot e^{x^4}\, dx$$

$\boxed{e^{x^4} = u \quad 4x^3 e^{x^4} dx = du}$

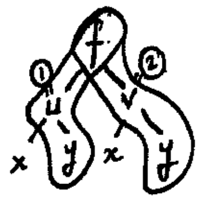
$$= \frac{1}{4} \int_1^{e^{16}} du = \frac{1}{4}\left(e^{16} - 1\right)$$



Q8. (6+6=12 pts) This question has unrelated parts.

a) Let $z = f(x^3 y, \sqrt{xy+1})$ be differentiable everywhere. If $f_1(0,1) = 1$, $f_2(0,1) = 2$, evaluate $\frac{\partial z}{\partial y}$ at the point $(x,y) = (1,0)$.

$$\frac{\partial z}{\partial y} = f_1(x^3 y, \sqrt{xy+1}) \cdot x^3 + f_2(x^3 y, \sqrt{xy+1}) \cdot \frac{x}{2\sqrt{xy+1}}$$



$$\Rightarrow \frac{\partial z}{\partial y}\Big|_{(1,0)} = f_1(0,1) \cdot 1^3 + f_2(0,1) \cdot \frac{1}{2} = 1 \cdot 1 + 2 \cdot \frac{1}{2} = 2$$

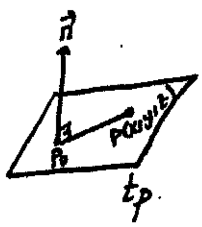
b) Find the equation of the tangent plane to the surface $z = \underbrace{x^3 \ln(y) + 3e^{xy}}_{f(x,y)}$ at the point $\underbrace{P_0 = (0,1)}_{(0,1,f(0,1)) = \tilde{P}_0}$.

Define $F(x,y,z) = f(x,y) - z$ s.t. $F = 0$ is a level surface.

$\nabla F(x,y,z) = \left(3x^2 \ln y + 3y e^{xy},\ \frac{x^3}{y} + 3x e^{xy},\ -1\right)$

$\nabla F(\tilde{P}_0)$ gives the normal to tangent plane at $\tilde{P}_0$.

$\vec{n} = \nabla F(0,1,3) = (3,0,-1)$. Then:



$t_p : 3(x-0) + 0 \cdot (y-1) - 1 \cdot (z-3) = 0$

$\Rightarrow \boxed{3x + 3 = z}$

P5

MATH-118

Final

-2023-

(Yaz okulu)

MATH 118 Exam 3 (100 points)
September 12, 2023 (09:00-10:20)

FULL NAME : KEY

Student ID : ____________________

No notes, calculators, or other aids are allowed. Read all directions carefully and write your answers in the space provided.

1. **Circle True (T) or False (F). No need for justification.**

   (a) (4 points) If $f_{xx}(a,b) > 0$ and $f_{yy}(a,b) > 0$ at a critical point (a,b), then (a,b) is a local minimum point of $f(x,y)$. (T) or (F)

   (b) (4 points) If $f_{xx}(a,b) < 0$ and $f_{yy}(a,b) > 0$ at a critical point (a,b), then (a,b) is a saddle point of $f(x,y)$. (T) or (F)

   (c) (4 points) If $\nabla f \neq 0$ for a differentiable function f at all points of its domain D, which is closed and bounded, then the absolute minimum and maximum values of f on D must occur on the boundary of D. (T) or (F)

   (d) (4 points) The directional derivative of a differentiable function $f(x,y)$ at (a,b) in the direction of the unit vector $\mathbf{j}$ is equal to $f_y(a,b)$. (T) or (F)

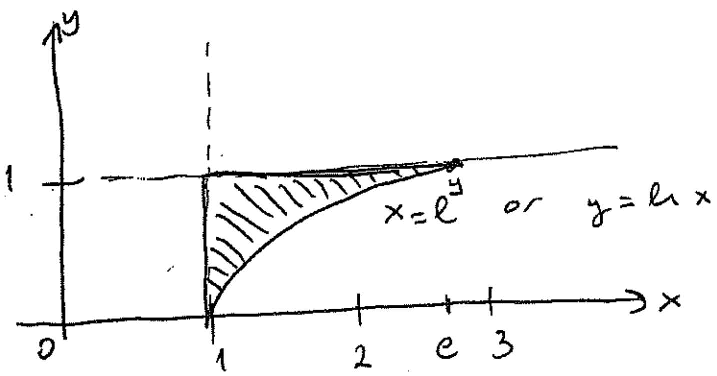
   (e) (4 points) If $\lim_{x\to 0} f(x,0) = \lim_{y\to 0} f(0,y) = L$, then $\lim_{(x,y)\to(0,0)} f(x,y) = L$. (T) or (F)

   (f) (4 points) If $\lim_{(x,y)\to(0,0)} f(x,y) = L$, then $\lim_{x\to 0} f(x,0) = \lim_{y\to 0} f(0,y) = L$. (T) or (F)

   (g) (4 points) If $R = [a,b]$ denotes an interval, then $\iint_R dA = b - a$. (T) or (F)

   (h) (4 points) $\iint_{x^2+y^2\le 1} \sin(xy)\, dA = 0$. (T) or (F)

2. (20 points) Given $\int_0^1 \int_1^{e^y} f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$, find a, b, c, and d. **SHOW** your work and write your results in the box.



$a =$ 1 $\quad b =$ e $\quad c =$ ln x $\quad d =$ 1

P6

KEY

3. The tangent plane to the graph of $f(x, y)$ at the point $(1, 2, -7)$ is given by $z = -3 + 2x - 3y$. Answer the following:

(a) (8 points) The gradient of $f(x, y)$ at $(1, 2)$ is

$\nabla f(1,2) = 2i - 3j$

(b) (8 points) The normal vector $\mathbf{n}$ to $z = f(x, y)$ at the point $(1, 2, -7)$ is

$\mathbf{n} = 2i - 3j - k$ (or any scalar multiple)

(c) (8 points) The directional derivative of f at $(1, 2)$ in the direction of the vector $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$ is

$D_{\mathbf{u}} f(1,2) = -1/5$

4. (24 points) Using the method of Lagrange multipliers, find the absolute maximum and minimum values of $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 8$.

$\nabla f = \lambda \cdot \nabla g$ $\quad (g(x,y) = 4x^2 + y^2)$

$$\left.\begin{aligned} y &= 8\lambda x \\ x &= 2\lambda y \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} y^2 &= 4x^2 \\ 4x^2 + y^2 &= 8 \end{aligned}\right\} \Longrightarrow \begin{aligned} x^2 &= 1 \\ x &= \mp 1 \\ y &= \mp 2 \end{aligned}$$

$(1,2), (1,-2), (-1,2), (-1,-2)$

maximum = 2 minimum = -2

MATH-118
Final
-2023-

METU Department of Mathematics

Math 118 Calculus II – E. Coşkun, M. Korkmaz, Ş. Öztürk – Final Exam, June 15, 2023, 09:30		
NAME, LAST NAME (CAPITAL letters only)	STUDENT NUMBER	DEPARTMENT/SECTION
6 QUESTIONS ON 6 PAGES, SHOW YOUR WORK, 120 MINUTES, TOTAL 100 PTS		

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of any unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rul· , the examination will result in immediate cancellation and disciplinary procedures. Your Signature :

Q.1 (8+6 pts) Let $f(x) = \sum_{n=1}^{\infty} \frac{5x^n}{4^{n-1}}$.

a) Find the interval of convergence of the series $f(x)$.

b) Compute $f'''(0)$

Q.2. (5+9 pts) The following parts are independent of each other.

a) Let R be the region in the first quadrant of the xy-plane bounded by the curve $y = \sqrt{x}$ and the line $y = x$. The solid S is generated by rotating the region R around the y-axis. Draw the region R and write an integral which computes the volume of S. (Do not compute the integral!)

b) Let S be the solid bounded above by the surface $z = -x^2 - y^2 + 6$ and having a triangular base in the xy-plane with vertices $(0,0)$, $(1,1)$, $(0,1)$. Find the volume of S.

Q.3 (8+4+6 pts) a) Write the parametric equations of the line in 3-space passing through the points $P=(1,-2,1)$, $Q=(4,-2,-2)$.

b) Find a normal vector for the plane containing the points $P=(1,-2,1)$, $Q=(4,-2,-2)$, and $R=(4,1,4)$.

, 3, 3

c) Find an equation for the plane containing the point $A=(1,0,2)$ which is parallel to the plane given by $3x+y-3z=118$.

Q.4 (7+4+7 pts) Let $f(x,y) = \ln(2x+y)$.

a) Find an equation of the tangent plane to the graph of $f(x,y)$ at the point $(-1,3,f(-1,3))$.

b) Find an approximate value for $f(-0.09, 3.02)$ using the tangent plane at $(-1,3,f(-1,3))$.

c) Determine if the function $f(x,y)$ is increasing or decreasing at the point $(-1,3)$ in the direction of the vector $\vec{u} = \langle 3,4\rangle = 3\mathrm{i}+4\mathrm{j}$.

P11

MATH-118

Final

-2022-

(Yaz okulu)

By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

METU Department of Mathematics

CALCULUS II, FINAL EXAM	Signature:
Code : MATH 118 Acad. Year: 2021-2022 Semester : Summer Instructor : M.Korkmaz, B.Korkmaz Date : September 03, 2022 Time : 09:00 Duration : 100 minutes	Full Name (USE CAPITAL LETTERS): SOLUTIONS Student ID:

1	2	3	4	5	6	6 questions on 4 pages TOTAL 100 POINTS		

1. (12+12 pts) Evaluate the following limits or show that they do not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4+y^4}$

Along $y=mx$, $\lim_{x\to 0} \frac{m^3x^4}{x^4+m^4x^4} = \frac{m^3}{1+m^4}$ depends on m.

So the limit does not exist.

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2\sin y}{x^2+y^4}$

$$0 \le \left|\frac{x^2\sin y}{x^2+y^4}\right| = \frac{x^2}{x^2+y^4}|\sin y| \le |\sin y| \quad \left(\frac{x^2}{x^2+y^4}\le 1\right)$$

$\lim_{(x,y)\to(0,0)} 0 = 0$, $\lim_{(x,y)\to(0,0)} |\sin y| = 0$.

By Squeeze Theorem

$$\lim_{(x,y)\to(0,0)} \frac{x^2\sin y}{x^2+y^4} = 0.$$

Full Name: Student ID:

2. (8+8+8 pts) Let $f(x,y)=\frac{x}{x^2+y^2}$.
(a) Find the gradient of $f(x,y)$ at the point $(1,2)$.

$f_x=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$, $f_x(1,2)=\frac{4-1}{25}=\frac{3}{25}$

$f_y=-\frac{2xy}{(x^2+y^2)^2}$, $f_y(1,2)=-\frac{4}{25}$ $\quad \vec{\nabla} f(1,2)=\langle\frac{3}{25},-\frac{4}{25}\rangle$

(b) Find an equation of the tangent plane to the graph of $f(x,y)$ at $(x,y)=(1,2)$.

$\vec{n}=\langle f_x(1,2), f_y(1,2), -1\rangle=\langle\frac{3}{25},-\frac{4}{25},-1\rangle$

$f(1,2)=\frac{1}{5}$

Tangent plane: $\frac{3}{25}(x-1)-\frac{4}{25}(y-2)-(z-\frac{1}{5})=0$

or $z=\frac{1}{5}+\frac{3}{25}(x-1)-\frac{4}{25}(y-2)$

(c) Find the rate of change of $f(x,y)$ at $(1,2)$ in the direction of $v=(5,12)$.

$D_{\vec{v}}f(1,2)=\frac{1}{|\vec{v}|}\vec{\nabla}f(1,2)\cdot\vec{v}$

$=\frac{1}{\sqrt{25+12^2}}\langle\frac{3}{25},-\frac{4}{25}\rangle\cdot\langle 5,12\rangle$

$=\frac{1}{25\sqrt{25+12^2}}(15-48)=-\frac{33}{25\sqrt{169}}$

3. (12 pts) Let $f(x,y)$ be a function whose first partial derivatives exist and continuous, and let $g(x,y)=f(x^2-y^2,2xe^y)$. If $f(3,4e)=6$, $f_1(3,4e)=7$ and $f_2(3,4e)=8$, find $g_x(2,1))$ and $g_y(2,1))$.

$g_x=2xf_1+2e^yf_2$

$g_x(2,1)=4f_1(3,4e)+2ef_2(3,4e)$

$=28+16e$

$g_y=-2yf_1+2xe^yf_2$

$g_y(2,1)=-2(7)+4e(8)=-14+32e$

4. (16 pts) Find the maximum and minimum values of $f(x,y) = e^{-xy}$ on the region $x^2 + 4y^2 \leq 2$.

Critical points: $f_x = -ye^{-xy} = 0 \Rightarrow y = 0$

$f_y = -xe^{-xy} = 0 \Rightarrow x = 0$

$(0,0)$ critical point. $\boxed{f(0,0) = 1}$

There are no singular points.

Boundary: $x^2 + 4y^2 = 2$, $g(x,y) = x^2 + 4y^2 - 2 = 0$

We can use Lagrange Multipliers:

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\left.\begin{aligned} -ye^{-xy} &= \lambda 2x \\ -xe^{-xy} &= \lambda 8y \\ x^2 + 4y^2 &= 2 \end{aligned}\right\}$$

$x = 0 \Rightarrow y = 0$ not possible.

$y = 0 \Rightarrow x = 0$ not possible.

So $x \neq 0$, $y \neq 0$ and

$$\lambda = \frac{-ye^{-xy}}{2x} = \frac{-xe^{-xy}}{8y}$$

$\Rightarrow 4y^2 = x^2$

use it in $x^2 + 4y^2 = 2 \Rightarrow 2x^2 = 2 \Rightarrow x = \mp 1$

$y^2 = \frac{1}{4} \Rightarrow y = \mp \frac{1}{2}$

$f(1, \frac{1}{2}) = f(-1, -\frac{1}{2}) = e^{-1/2} = \frac{1}{\sqrt{e}} < 1$

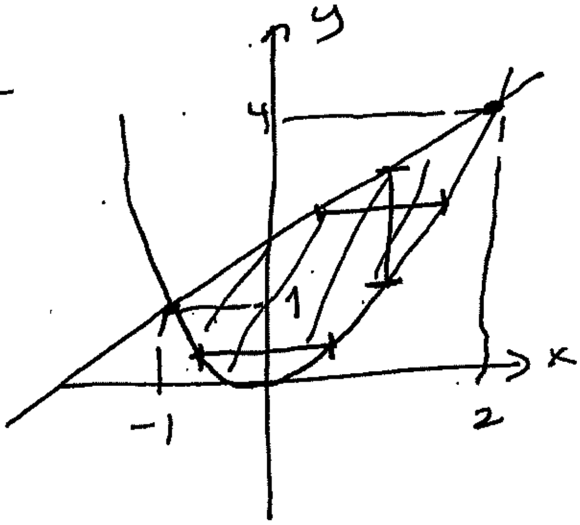
$f(1, -\frac{1}{2}) = f(-1, \frac{1}{2}) = e^{1/2} = \sqrt{e} > 1$

So $\sqrt{e}$ is maximum value

$\frac{1}{\sqrt{e}}$ is minimum value

5. (8+8 pts) Let D be the region bounded by the parabola $y = x^2$ and the line $y = x + 2$. Write the double integral $\iint_D f(x,y)\,dA$ as iterated integrals in the order

(a) $dy\,dx$



Intersection points: $x^2 = x+2$

$\Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = 2, -1$

$-1 \le x \le 2, \quad x^2 \le y \le x+2$

$$I = \int_{-1}^{2} \int_{x^2}^{x+2} f(x,y)\,dy\,dx$$

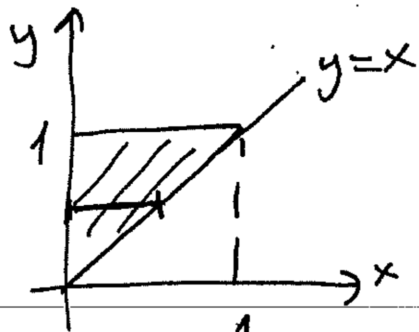
(b) $dx\,dy$.

$0 \le y \le 1 , \quad 1 \le y \le 4$

$-\sqrt{y} \le x \le \sqrt{y}, \quad y-2 \le x \le \sqrt{y}$

$$I = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} f(x,y)\,dx\,dy$$

6. (8 pts) Evaluate the integral $\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx$.



$$= \int_0^1 \int_0^y \cos(y^2)\,dx\,dy$$

$$= \int_0^1 \cos(y^2)\, x \Big|_{x=0}^{y} dy$$

$$= \int_0^1 \cos(y^2)\, y\, dy$$

$$= \frac{1}{2} \sin(y^2) \Big|_0^1 = \frac{1}{2}\sin 1$$

MATH-118
Final
-2022-

METU Department of Mathematics

<table>
<tr><td colspan="3">Calculus II: Final</td></tr>
<tr><td>Code : MATH 118
Acad. Year : 2021-2022
Semester : Spring
Instructors : E.C, S.Ö, Ö.Y.
Date : June 23, 2022
Time : 09:30
Duration : 100 minutes</td><td colspan="2">Full Name (USE CAPITAL LETTERS): Solutions
Student ID:</td></tr>
<tr><td>5 QUESTIONS ON 4 PAGES
TOTAL 100 POINTS</td><td>SHOW
YOUR WORK</td><td>Signature:</td></tr>
</table>

By signing above, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Q.1 (12+8 pts) Let $f(x) = \sum_{n=1}^{\infty} \frac{n(2x+1)^n}{(n^2+1)3^n}$. $=: a_n$

a) Find the interval of convergence I of the series $f(x)$.

Ratio Test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)(2x+1)^{n+1}}{[(n+1)^2+1]3^{n+1}} \frac{3^n(n^2+1)}{n(2x+1)^n} \right| = \frac{|2x+1|}{3} < 1$

$\Rightarrow -3 < 2x+1 < 3 \Rightarrow -2 < x < 1$

at $x=-2$, $\sum_{n=1}^{\infty} \frac{n(-1)^n}{(n^2+1)} = b_n$: ① $b_n \cdot b_{n+1} < 0$ ② $|b_n| = \frac{n}{n^2+1} \downarrow 0$

Thus, by AST, $\sum_{n=1}^{\infty} b_n$ is convergent.

at $x=1$, $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent by L.C.T with $\sum_{n=1}^{\infty} \frac{1}{n}$.

Hence, interval of convergence of $f(x)$ is $I = [-2, 1)$.

b) Compute $f^{(118)}(\frac{-1}{2})$. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1/2)}{n!}(x+1/2)^n = \sum_{n=1}^{\infty} \frac{n 2^n (x+1/2)^n}{(n^2+1)3^n}$

$\Rightarrow \frac{f^{(118)}(-1/2)}{(118)!} = \frac{118 \cdot 2^{118}}{(118^2+1)3^{118}} \Rightarrow f^{(118)}(-1/2) = \frac{(118)!\, 118\, 2^{118}}{(118^2+1)\, 3^{118}}$.

Q.2 (10+10 pts) Evaluate the following integrals.

a) $\int_0^\infty xe^{-x}dx. = \lim_{t\to\infty} \int_0^t xe^{-x}\,dx$

$$\int_0^t \underset{u}{x}\,\underbrace{e^{-x}\,dx}_{dv} = -x\cdot e^{-x}\Big|_{x=0}^{t} + \int_0^t e^{-x}dx = -te^{-t} + 1 - e^{-t}$$

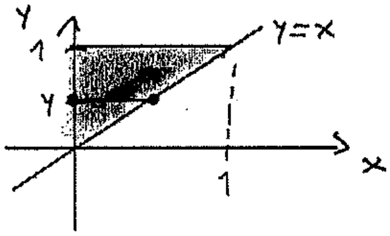
integratration by part

$$\lim_{t\to\infty}\frac{t}{e^t} \underset{L\cdot H}{=} \lim_{t\to\infty}\frac{1}{e^t} = 0 \quad \text{and} \quad \lim_{t\to\infty} e^{-t} = 0. \text{ Thus}$$

$$\int_0^\infty xe^{-x}\,dx = 1.$$

b) $\int_0^1\int_x^1 e^{x/y}dydx$

Domain: $x \leq y \leq 1$, $0 \leq x \leq 1$.



Change the order : $0 \leq y \leq 1$, $0 \leq x \leq y$

$$\int_0^1\int_x^1 e^{x/y}\,dy\,dx = \int_0^1\int_0^y e^{x/y}\,dx\,dy = \int_0^1 \left(y\cdot e^{x/y}\right)\Big|_{x=0}^{y} dy$$

$$= \int_0^1 (ey - y)\,dy = (e-1)\frac{1}{2}.$$

Full Name: **Student ID:**

Q.3 (10+10 pts) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

a) Is f continuous at $(0,0)$?

$$0 \leq \left| \frac{x^3 - y^3}{x^2+y^2} \right| \leq \frac{|x^3| + |y^3|}{x^2+y^2} = \frac{|x|x^2 + |y|y^2}{x^2+y^2} \leq |x| + |y| \longrightarrow 0 \text{ as } (x,y) \to 0$$

Thus $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ by Squeeze Theorem and f is continuous at $(0,0)$.

b) Find $\dfrac{\partial f}{\partial x}(0,0)$. $= \lim_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \dfrac{x}{x} = 1.$

Q.4 (10+10 pts)

a) Find the equation of the tangent plane to the surface defined by $\underbrace{x^3 + y^3 + z^3 = 3x^2y^2z^2}$ at $(1,1,1)$.

$$F(x,y,z) = x^3 + y^3 + z^3 - 3x^2y^2z^2 = 0$$

$$\vec{n} \parallel \nabla F(1,1,1) = \left\langle 3x^2 - 6xy^2z^2,\ 3y^2 - 6yx^2z^2,\ 3z^2 - 6zx^2y^2 \right\rangle \Big|_{(1,1,1)}$$

$$= \langle -3,-3,-3 \rangle \parallel \langle -1,-1,-1 \rangle$$

Tangent plane equation: $\langle -1,-1,-1 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$

$\Rightarrow -x - y - z + 3 = 0$ or $x + y + z = 3$.

b) Find the directional derivative of $f(x,y) = \sin(xy) + xe^{x^2+y^2}$ at the point $(1,\pi)$ in the direction $u = < 2, 3 >$

$$\tilde{u} = \frac{u}{|u|} = \frac{\langle 2,3 \rangle}{\sqrt{13}}, \quad \nabla f(1,\pi) = \left\langle \cos(xy)y + e^{x^2+y^2} + xe^{x^2+y^2} \cdot 2x,\ \cos(xy)x + e^{x^2+y^2} \cdot 2xy \right\rangle \Big|_{(1,\pi)}$$

$$= \left\langle -\pi + 3e^{\pi^2+1},\ -1 + 2\pi e^{\pi^2+1} \right\rangle$$

$$D_{\tilde{u}} f(1,\pi) = \nabla f \cdot \tilde{u} = \frac{1}{\sqrt{13}} \left[-2\pi + 6e^{\pi^2+1} \quad -3 + 6\pi e^{\pi^2+1} \right]$$

Q.5 (10+10 pts)

a) Find and classify critical point(s) of $f(x,y) = x^2 + y^2 + 4x - 4y$ in the disc defined by $x^2 + y^2 < 9$.

$f_x(x,y) = 2x + 4 = 0$, $f_y(x,y) = 2y - 4 = 0 \Rightarrow x = -2, y = 2$.

The only critical point is $(2,2)$.

$f_{xx}(2,2) = 2$, $f_{yy}(2,2) = 2$, $f_{xy}(2,2) = 0$

$D(x,y) = f_{xx}(2,2) f_{yy}(2,2) - f_{xy}^2(2,2) = 4 > 0$ and $f_{xx}(2,2) = 2 > 0$.

Thus f has a local minimum at $(2,2)$.

b) Use Lagrange multipliers to find the maximum and minimum values of $f(x,y) = x^2 + y^2 + 4x - 4y$ on the circle $x^2 + y^2 = 9$. $g(x,y) = x^2 + y^2 - 9$

Solve $\nabla f = \langle 2x+4, 2y-4 \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle$

and $g(x,y) = 0$.

① $2x + 4 = 2\lambda x$ ② $2y - 4 = 2\lambda y$ ③ $x^2 + y^2 - 9 = 0$

$\Rightarrow \lambda = \frac{x+2}{x} = \frac{y-2}{y} \Rightarrow y = -x \Rightarrow 2x^2 = 9 \Rightarrow x = \mp \frac{3}{\sqrt{2}}$

Solutions: $\left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right), \left(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

$f\left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right) = \frac{9}{2} + \frac{9}{2} + 4 \cdot \frac{6}{\sqrt{2}} = 9 + 12\sqrt{2}$ $\longrightarrow$ abs-maximum

$f\left(\frac{-3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = 9 - 4 \cdot \frac{6}{\sqrt{2}} = 9 - 12\sqrt{2}$. $\longrightarrow$ abs. minimum

MATH-118
final
-2019-
(Yaz Okulu)

M E T U Department of Mathematics

Math 118 Calculus II Exam 3 08.08.2019		
Last Name : Name : Student No:	Section : Time : 09: 00 Duration : 120 *minutes*	Signature
6 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS

1	2	3	4	5	6			

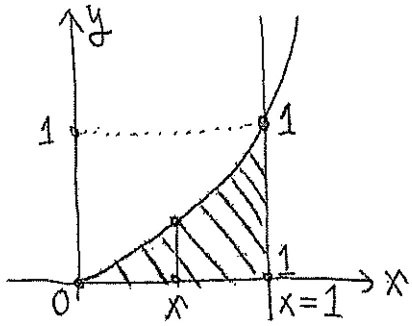
1. (7+6=13 points) This problem has two unrelated parts about double integrals.

(a) Evaluate the integral $I = \iint_R \frac{y}{x^5+1}\, dA$ where $R = \{(x,y) \mid 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$

$$I = \int_0^1 \int_0^{x^2} \frac{y}{x^5+1}\, dy\, dx = \int_0^1 \left(\frac{1}{2} \frac{y^2}{(x^5+1)} \Big|_0^{x^2} \right) dx$$

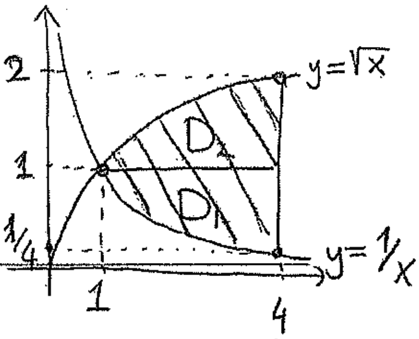
$$= \frac{1}{2} \int_0^1 \frac{x^4}{x^5+1}\, dx = \frac{1}{10} \int_1^2 \frac{du}{u} = \frac{1}{10} \ln|u| \Big|_1^2 = \frac{\ln 2}{10}$$

$u = x^5+1$
$du = 5x^4\, dx$



(b) Rewrite the iterated integral $I = \int_1^4 \int_{\frac{1}{x}}^{\sqrt{x}} f(x,y)\, dy\, dx$ in $dx\, dy$ order. Do not evaluate the integral.

$$I = \underbrace{\int_{1/4}^{1} \int_{1/y}^{4} f(x,y)\, dx\, dy}_{\text{on } D_1} + \underbrace{\int_1^2 \int_{y^2}^{4} f(x,y)\, dx\, dy}_{\text{on } D_2}$$



2. (12 points) Find and classify the critical points of $f(x,y) = -2x^2 + 4xy - y^4$.

$$\left.\begin{array}{l} f_x(x,y) = -4x + 4y = 0 \Rightarrow x = y \\ f_y(x,y) = 4x - 4y^3 = 0 \Rightarrow x = y^3 \end{array}\right\} \Rightarrow y = y^3 \Rightarrow y(y^2-1) = 0 \quad \begin{array}{l} y=0 \\ y=+1 \\ y=-1 \end{array}$$

Critical points are $(0,0), (1,1), (-1,-1)$

$D(a,b) = f_{xx}(a,b) \cdot f_{yy}(a,b) - f_{xy}(a,b) \cdot f_{yx}(a,b)$

$f_{xx}(x,y) = -4$
$f_{yy}(x,y) = -12y^2$
$f_{xy}(x,y) = f_{yx}(x,y) = 4$

$(0,0)$	$(1,1)$	$(-1,-1)$
$D(0,0) = -16 < 0$	$D(1,1) = 32 > 0$	$D(-1,-1) = 32 > 0$
Saddle Point	$f_{xx}(1,1) = -4 < 0$	$f_{xx}(-1,-1) = -4 < 0$
	Local Maximum	Local Maximum

3. **(8+7=15 points)** This problem has two unrelated parts about limits.

(a) Show that $\lim_{(x,y)\to(0,0)} \frac{x^3 \sin(x^2+y^2)}{x^2+y^4} = 0$

$$0 \leq \left|\frac{x^3 \cdot \sin(x^2+y^2)}{x^2+y^4}\right| \leq \left|\boxed{\frac{x^2}{x^2+y^4}} \cdot x\right| \leq |x|$$

as $|\sin(x^2+y^2)| \leq 1$

$x^2 \leq x^2+y^4$

by Squeeze Thm. $\to 0$

$\to 0$ as $(x,y)\to(0,0)$

(b) Does the limit $\lim_{(x,y)\to(0,0)} \frac{x^3y^5}{x^3+y^8-x^2y}$ exist? **Explain your answer.**

Over $x=0$ $\quad \lim_{y\to 0} \frac{0\cdot y^5}{0+y^8-0} = \lim_{y\to 0} 0 = \underline{\underline{0}}$

Over $y=x$ $\quad \lim_{x\to 0} \frac{x^3\cdot x^5}{x^3+x^8-x^2\cdot x} = \lim_{x\to 0} \frac{x^8}{x^8} = \lim_{x\to 0} 1 = \underline{\underline{1}}$

Since $1\neq 0$, limit does not exist.

4. **(8+7=15 points)** Given $f(x,y) = \begin{cases} \frac{(x-1)\sin(x-y)}{x-y}, & \text{if } y \neq x \\ 0, & \text{if } y = x \end{cases}$

(a) Compute $f_1(1,1)(= f_x(1,1))$.

$$f_1(1,1) = \lim_{h\to 0} \frac{f(1+h,1)-f(1,1)}{h} = \lim_{h\to 0} \frac{\frac{(1+h-1)\cdot\sin(1+h-1)}{(1+h)-1} - 0}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\cdot \sin(h)}{h\cdot \cancel{h}} = \lim_{h\to 0} \frac{\sin(h)}{h} = \boxed{1}$$

(b) Compute $f_2(0,\pi)(= f_y(0,\pi))$

$(0,\pi)$ is not on $y=x$.

$$f_2(x,y) = (x-1)\cdot\left[\frac{\cos(x-y)\cdot(-1)(x-y) - (-1)\cdot\sin(x-y)}{(x-y)^2}\right], \quad y\neq x$$

$$f_2(0,\pi) = (0-1)\cdot\left(\frac{\cos(0-\pi)\cdot(-1)\cdot(0-\pi) - (-1)\cdot\sin(0-\pi)}{(0-\pi)^2}\right)$$

$$= \frac{\pi}{\pi^2} = \boxed{\frac{1}{\pi}}$$

5. (10+10=20 points) (a) Find the absolute extreme values of $f(x,y) = x^2 - y^2$ on $D = \{(x,y) \mid x^2+y^2=1\}$ by using Lagrange multipliers method.

Let $g(x,y) = x^2+y^2-1$ to use Lagrange multipliers method.

i) $\nabla f = \lambda \cdot \nabla g \Rightarrow$ 1) $2x = \lambda 2x$ i.e. $2x(\lambda-1)=0$

2) $-2y = \lambda \cdot 2y$ i.e. $2y(\lambda+1)=0$

ii) $g(x,y)=0 \Rightarrow$ 3) $x^2+y^2=1$

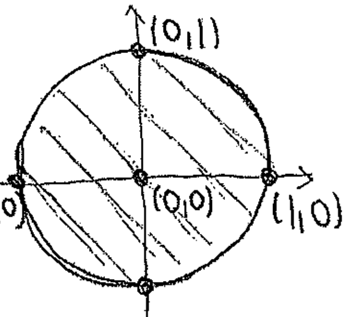
1) $x=0$	1) $\lambda=1$
3) $y=\mp 1$	2) $y=0$
2) $\lambda=-1$	3) $x=\mp 1$
$(0,1)$, $(0,-1)$	$(1,0)$, $(-1,0)$

$f(1,0)=f(-1,0)=1$ Absolute Maximum Value

$f(0,1)=f(0,-1)=-1$ Absolute Minimum Value

(b) Find the absolute extreme values of $f(x,y) = x^2 - y^2$ on the domain $R = \{(x,y) \mid x^2+y^2 \le 1\}$.

f is continuous on closed & bounded domain R. Hence, by Extreme Value Thm. f has the absolute extreme values on R



i) Critical pts: $\left.\begin{aligned} f_x(x,y) &= 2x = 0 \\ f_y(x,y) &= -2y = 0 \end{aligned}\right\} \Rightarrow (0,0)$ is the only critical point in R.

and $f(0,0)=0$

ii) No singular points

iii) Boundary points. It's done in part (a).

Hence, f has the absolute maximum value 1 on R.

" " minimum " -1

6. (5+6+6+8=25 points) Let $f(s,t)$ be a differentiable function with the following values at the points $(1,0)$ and $(1,1)$.

	f	f_s	f_t
(1,0)	1	-2	4
(1,1)	4	1	-2

(a) Write the tangent plane equation to $z = f(s,t)$ at $(1,0,1)$.

$$z = f(1,0) + f_s(1,0)\cdot(s-1) + f_t(1,0)\cdot(t-0)$$
$$= 1 - 2\cdot(s-1) + 4\cdot t = \underline{-2s + 4t + 3}$$

(b) Find the directional derivative of $f(s,t)$ at $(1,1)$ in the directon of the vector $\vec{u} = \langle 1, -1\rangle$.

$\frac{\vec{u}}{|\vec{u}|} = \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$. Since f is differentiable

$$D_{\vec{u}} f(1,1) = D_{\frac{\vec{u}}{|\vec{u}|}} f(1,1) = \nabla f(1,1) \cdot \frac{\vec{u}}{|\vec{u}|} = \langle 1, -2\rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle = \boxed{\frac{3}{\sqrt{2}}}$$

(c) Let $g(u_1, u_2)$ be the function defined on $D = \{(u_1,u_2) \mid u_1^2 + u_2^2 = 1\}$ as

$$g(u_1,u_2) = \lim_{h \to 0+} \frac{f(1 + hu_1, 1 + hu_2) - 4}{h}$$

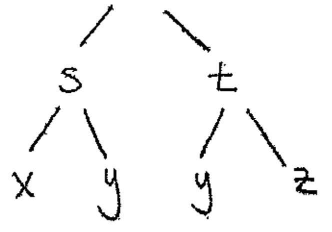
Find the maximum and minimum values of $g(u_1, u_2)$, if exist.

$g(u_1, u_2) = D_{\vec{u}} f(1,1)$, where $\vec{u} = \langle u_1, u_2 \rangle$. Hence,

the maximum value of $g(u_1,u_2)$ is equal to $|\nabla f(1,1)| = \sqrt{5}$

the minimum value of 〃 〃 $-|\nabla f(1,1)| = -\sqrt{5}$.

(d) Let $h(x,y,z) = f(x^2 + 2y, e^y z)$. Compute $\frac{\partial h}{\partial y}(1,0,1)$.



$x=1$, $y=0$, $z=1$
$s=1$, $t=1$

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y} = f_1 \cdot 2 + f_2 \cdot e^y z$$

$$\frac{\partial h}{\partial y}(1,0,1) = f_1(1,1)\cdot 2 + f_2(1,1)\cdot e^0 1$$
$$= 1\cdot 2 + (-2)\cdot 1 = \boxed{0}$$

MATH-128
final
-2019-

METU
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	CALCULUS 2 Final	

Code : *Math 118* Acad. Year : *2018-2019* Semester : *Spring* Coordinator : *H.T./H.Ö./E.C.* Date : *May 24, 2019* Time : *9:30* Duration : *120 minutes*	Last Name : Name : Student No. : Department : Section : Signature : 7 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

Q1	Q2-3	Q4	P4	SHOW YOUR WORK	

Q.1 ($4\times5 = 20$ pts) Let $\sum_{n=1}^{\infty} a_n$ be a series whose N−th partial sum is given by

$$S_N = \frac{2N^2}{N^2+3N+1}.$$

(a) Is the series $\sum_{n=1}^{\infty} a_n$ convergent? If it is possible, find its sum.

YES, $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$

$= \lim_{N\to\infty} \frac{2N^2}{N^2+2N+1} = 2.$

(b) Is the sequence $\{a_n\}$ convergent? YES.

Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$.

(c) Let $\{b_n\}$ be another sequence, where $b_n = e^{a_n}$. Is the sequence $\{b_n\}$ convergent?

YES, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} e^{a_n}$

$= e^{\lim_{n\to\infty} a_n} = e^0 = 1.$

(d) Is the series $\sum_{n=1}^{\infty} b_n$ convergent?

No, $\lim_{n\to\infty} b_n = 1 \neq 0.$

Q.2 (10 pts) Show that the function $f(x,y) = \begin{cases} \frac{xy^3}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is continuous at the point $(x,y) = (0,0)$.

Note that, $0 \le \left| \frac{xy^3}{x^4+y^2} \right| = \left| \frac{xy\, y^2}{x^4+y^2} \right| \le |xy|$

0 as $(x,y) \to (0,0)$; 0 as $(x,y) \to (0,0)$

Hence, by squeeze theorem $\lim_{(x,y)\to(0,0)} \left| \frac{xy^3}{x^4+y^2} \right| = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^4+y^2} = 0$

Also, $f(0,0) = 0 = \lim_{(x,y)\to(0,0)} f(x,y) \Rightarrow f$ is cont. at $(0,0)$.

Q.3 (10 pts) Let ℓ be a line in $\mathbb{R}^3$ given by $\frac{x-1}{-1} = \frac{y}{3} = \frac{z-1}{2}$. Find the derivative of the function $f(x,y,z) = e^{x^2+y^2-z^2}$ at the point $(x,y,z) = (1,0,1)$ in the direction of line ℓ.

$\frac{x-1}{-1} = \frac{y}{3} = \frac{z-1}{2} = t \Rightarrow x = 1-t,\ y = 3t,\ z = 1+2t$

direction vector of ℓ is $\langle -1, 3, 2 \rangle = \vec{V}$. Let $\vec{u} = \frac{\vec{V}}{\|\vec{V}\|}$.

So, $\vec{u} = \langle \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \rangle$. Since f is differentiable,

$D_{\vec{u}} f(1,0,1) = \nabla f(1,0,1) \cdot \vec{u} = \vec{u} \cdot \langle 2xe^{x^2+y^2-z^2}, 2ye^{x^2+y^2-z^2}, -2ze^{x^2+y^2-z^2} \rangle \Big|_{(1,0,1)}$

$= \langle \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \rangle \cdot \langle 2, 0, -2 \rangle = \frac{-2}{\sqrt{14}} + 0 - \frac{4}{\sqrt{14}} = \frac{-6}{\sqrt{14}}$

Q.4 (3×10 = 30 pts) Consider the function $f(x,y) = \frac{x^3}{3} + \frac{x^2}{2} + \frac{y^2}{2} + y + 1$.

(a) Find and classify all the critical points of f.

$0 = f_x = x^2 + x \rightarrow x = 0,\ x = -1$; $A = f_{xx} = 2x+1$

$0 = f_y = y + 1 \rightarrow y = -1$; $B = f_{xy} = 0$

$C = f_{yy} = 1$.

$(0,-1)$:

$A = 1$

$B = 0$

$C = 1$

$B^2 - 4AC = -4$

$A > 0$

(local minimum)

$(-1,-1)$:

$A = -1$

$B = 0$

$C = 1$

$B^2 - 4AC = 4 > 0$

(saddle point).

(b) Use the method of Lagrange multipliers to find the maximum and minimum values of f on the curve (circle) $x^2+y^2=2$.

$$g(x,y)=x^2+y^2-2$$

$$\nabla f=(x^2+x)\,\hat{i}+(y+1)\,\hat{j}$$

$$\nabla g=(2x)\,\hat{i}+(2y)\,\hat{j}$$

$$\nabla f=\lambda\nabla g \Longrightarrow \begin{cases} x^2+x=2\lambda x \\ y+1=2\lambda y \end{cases}$$

$x=0, \quad y=\mp\sqrt{2}$

$x\neq 0, \quad \lambda=\frac{x+1}{2}$

$2\lambda=x+1$

$y+1=2\lambda y: \quad y+1=(x+1)y$

$1=xy \quad \therefore y=\frac{1}{x}$

$x^2+\frac{1}{x^2}=2 \qquad (x^2)^2-2(x^2)+1=0$

$x^2=1$

$x=\mp 1 \qquad (1,1), (-1,-1)$

(c) Consider the function f on the domain $\mathcal{D}=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 2\}$. Explain why f has both absolute maximum and minimum values on $\mathcal{D}$, and then find them.

boundary of D: $x^2+y^2=2$ in part (b).

critical points are in D.

f continuous on a closed and bounded region so it has extre

$f(0,-1)=\frac{1}{2}=0.5$

$$\boxed{\begin{array}{l}\text{max}: 2+\sqrt{2}\\ \text{min}: 0.5\end{array}}$$

$f(-1,-1)=-\frac{1}{3}+\frac{1}{2}+\frac{1}{2}\cancel{-1}\cancel{+1}=\frac{2}{3}=0.\overline{6}$

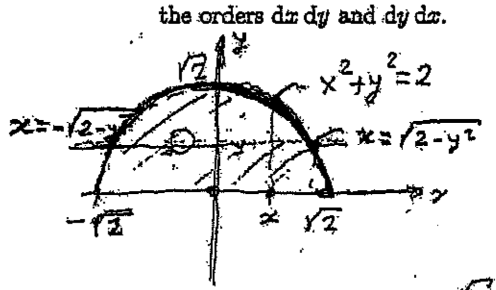
$f(1,1)=\frac{1}{3}+\frac{1}{2}+\frac{1}{2}+1+1=\frac{10}{3}\cong 3.\overline{3}$

$f(0,\sqrt{2})=1+\sqrt{2}+1=2+\sqrt{2}\cong 3.4\ldots$

$f(0,-\sqrt{2})=1-\sqrt{2}+1=2-\sqrt{2}\cong 0.59\ldots$

P26

Q.5 (10 pts) Let $\mathcal{D}$ be the semi-circular region bounded by the circle $x^2+y^2=2$ in the upper part of the xy−plane, where $y \geq 0$. Express the double integral $\iint_{\mathcal{D}} f(x,y)\,dA$ as iterated integrals in the orders $dx\,dy$ and $dy\,dx$.



$$D=\{(x,y)\in\mathbb{R}^2 : -\sqrt{2}\leq x\leq\sqrt{2}\ ,\ 0\leq y\leq\sqrt{2-x^2}\}$$

or

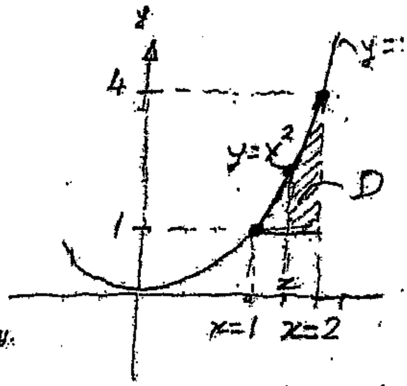
$$D=\{(x,y)\in\mathbb{R}^2 : -\sqrt{2-y^2}\leq x\leq\sqrt{2-y^2}\ ,\ 0\leq y\leq\sqrt{2}\}$$

$$\iint_D f(x,y)\,dA=\int_{-\sqrt{2}}^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} f(x,y)\,dy\,dx=\int_0^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x,y)\,dx\,dy$$

Q.6 (10 pts) Evaluate the iterated double integral $I=\int_0^1\int_{e^x}^{e^{2x}}\frac{x^2}{y}\,dy\,dx$.

$$I=\int_0^1 x^2\left[\ln y\Big|_{y=e^x}^{e^{2x}}\right]dx=\int_0^1 x^2(\ln e^{2x}-\ln e^x)\,dx=\int_0^1 x^2(2x-x)\,dx$$

$$I=\int_0^1 x^3\,dx=\frac{x^4}{4}\Big|_0^1=\frac{1}{4}$$



Q.7 (10 pts) Evaluate the iterated double integral $I=\int_1^4\int_{\sqrt{y}}^2 2y\cos(x^5-5x)\,dx\,dy$.

Changing the order, i.e. $D=\{(x,y)\in\mathbb{R}^2 : 1\leq x\leq 2,\ 1\leq y\leq x^2\}$, we have

$$I=\int_1^2\int_1^{x^2} 2y\cos(x^5-5x)\,dy\,dx=\int_1^2\cos(x^5-5x)\cdot y^2\Big|_1^{x^2}dx$$

$$=\int_1^2 (x^4-1)\cos(x^5-5x)\,dx \qquad \text{Let } u=x^5-5x\ ,\ u:-4\to 22$$

$$du=5(x^4-1)\,dx$$

$$=\frac{1}{5}\int_{-4}^{22}\cos(u)\,du=\frac{1}{5}\sin(u)\Big|_{-4}^{22}=\frac{1}{5}[\sin(22)-\sin(-4)]$$

$$I=\frac{1}{5}[\sin(22)+\sin(4)]$$

MATH-118 27

final

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METU
Department of Mathematics

CALCULUS II Final Exam	
Code : *Math 118* Acad. Year : *2017-2018* Semester : *Spring* Coordinator : *Fırat Arıkan* Date : *26.May.2018* Time : *9:30* Duration : *120 minutes*	Last Name : SOLUTION KEY Name : Student No. : Department : Section : Signature : SOLUTION KEY 6 QUESTIONS ON 4 PAGES TOTAL 100 POINTS

1	2	3	4	5	6	SHOW YOUR ORGANIZED WORK	

Q.1 (12+10 pts) (a) Compute $\int_2^\infty \frac{x+1}{x(x-1)^2}\,dx$ if it exists.

$$\frac{x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

$$\Rightarrow \quad x+1 = A(x^2-2x+1) + B(x^2-x) + Cx = (A+B)x^2 + (-2A-B+C)x + A$$

$$\Rightarrow \quad \left.\begin{array}{r} A+B=0 \\ -2A-B+C=1 \\ A=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=1 \\ B=-1 \\ C=2 \end{array}$$

$$\int_2^\infty \frac{x+1}{x(x-1)^2}dx = \lim_{R\to\infty}\int_2^R \left(\frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2}\right)dx$$

$$= \lim_{R\to\infty}\left(\ln|x| - \ln|x-1| - \frac{2}{x-1}\right)\Big|_2^R$$

$$= \lim_{R\to\infty}\left(\left(\ln\left|\frac{R}{R-1}\right| - \frac{2}{R-1}\right) - (\ln 2 - 2)\right)$$

$$= 2 - \ln 2$$

Answer $\boxed{2-\ln 2}$

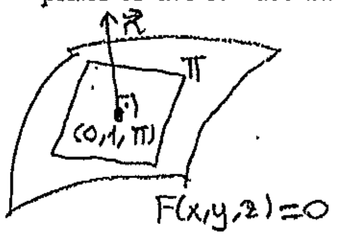
(b) <u>Using part (a)</u>, determine if $\sum_{n=2}^\infty \frac{1+n}{n(n-1)^2}$ is convergent or not.

Let $f(x) = \frac{1+x}{x(x-1)^2}$ so that $f(n) = \frac{1+n}{n(n-1)^2}$ $\forall\, n = 2,3,\ldots$

Since f is positive, continuous and nondecreasing on $[2,\infty)$ and $\int_2^\infty f(x)dx$ is convergent by part (a), so is $\sum_{n=2}^\infty \frac{1+n}{n(n-1)^2}$ by the integral test.

Answer $\boxed{\text{Convergent}}$

P28 Q.2 (14 pts) Using the fact that $F(x,y,z) = xz + y^2 + \sin(zy) - 1 = 0$ determines z as a function of x and y around the point $(0,1,\pi)$, write the equation of the tangent plane to the surface $xz + y^2 + \sin(zy) = 1$ at the point $(0,1,\pi)$.



$$\left.\begin{aligned} F_x &= z \\ F_y &= 2y + \cos(zy)\cdot z \\ F_z &= x + \cos(zy)\cdot y \end{aligned}\right\} \Rightarrow \begin{aligned} F_x(0,1,\pi) &= \pi \\ F_y(0,1,\pi) &= 2-\pi \\ F_z(0,1,\pi) &= -1 \end{aligned}$$

$$\vec{n} = \langle F_x(0,1,\pi), F_y(0,1,\pi), F_z(0,1,\pi)\rangle = \langle \pi, 2-\pi, -1\rangle.$$

$$\pi : \langle x-0, y-1, z-\pi\rangle \cdot \langle \pi, 2-\pi, -1\rangle = 0$$

$$\pi(x-0) + (2-\pi)(y-1) + (-1)(z-\pi) = 0$$

$$\pi x + (2-\pi)y - z = 2-2\pi$$

Answer

$$\boxed{\pi x + (2-\pi)y - z = 2-2\pi}$$

Q.3 (14 pts) Suppose that $f(x,y) = F(xy, \ln(x+y))$ where F is a differentiable function whose first partial derivatives satisfy

$$F_1(0,1) = 3, \quad F_2(0,1) = 5.$$

Compute the rate of change of f in the direction of the vector $\langle 1,-1\rangle$ at the point $(0,e)$. (Here e denotes the natural logarithm base.)

Let $\vec{u} = \langle 1,-1\rangle$. Then $|\vec{u}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

$$f_x = F_1(xy, \ln(x+y))\cdot y + F_2(xy, \ln(x+y))\cdot \frac{1}{x+y}$$

$$f_y = F_1(xy, \ln(x+y))\cdot x + F_2(xy, \ln(x+y))\cdot \frac{1}{x+y}$$

$$f_x(0,e) = F_1(0,1)\cdot e + F_2(0,1)\cdot \frac{1}{e} = 3e + \frac{5}{e}$$

$$f_y(0,e) = F_1(0,1)\cdot 0 + F_2(0,1)\cdot \frac{1}{e} = \frac{5}{e}$$

$$D_{\frac{\vec{u}}{|\vec{u}|}} f(0,e) = \vec{\nabla} f(0,e)\cdot \frac{\vec{u}}{|\vec{u}|} = \langle f_x(0,e), f_y(0,e)\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\rangle$$

$$= \left\langle 3e + \frac{5}{e}, \frac{5}{e}\right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right\rangle$$

$$= \left(3e + \frac{5}{e}\right)\frac{1}{\sqrt{2}} + \frac{5}{e}\left(\frac{-1}{\sqrt{2}}\right)$$

$$= \frac{3e}{\sqrt{2}}$$

Answer

$$\boxed{3e/\sqrt{2}}$$

Q.4 (10 + 12 pts) Consider the function $f(x,y) = x^2 + x + 2y^2$.

(a) Find and classify all critical points of f.

$$\left.\begin{aligned} 0 &= f_x = 2x+1 \\ 0 &= f_y = 4y \end{aligned}\right\} \Rightarrow \begin{aligned} x &= -\tfrac{1}{2} \\ y &= 0 \end{aligned}$$

The only critical point is $(-\frac{1}{2}, 0)$.

$A = f_{xx} = 2$

$B = f_{xy} = f_{yx} = 0$

$C = f_{yy} = 4$

At $(-\frac{1}{2},0)$: $\left.\begin{aligned} A &= 2 > 0 \\ B^2 - AC &= -8 < 0 \end{aligned}\right\}$

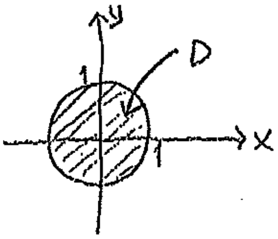
$\Rightarrow$ f has a local minimum value at $(-\frac{1}{2}, 0)$.

Answer

critical points: $(-\frac{1}{2}, 0)$.
local min value at $(-\frac{1}{2}, 0)$.

(b) **Using the method of Lagrange multipliers**, find the absolute maximum and the absolute minimum of f on the disk

$$D = \{(x,y) \mid x^2 + y^2 \le 1\}.$$



For the interior of D: critical pts: $(-\frac{1}{2}, 0)$.
singular pts: None.

For the boundary of D: ∂D: $x^2+y^2=1$.

Set $g(x,y) = x^2+y^2-1$.

The Lagrangian function: $L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$

$$\left.\begin{aligned} 0 &= L_x = f_x + \lambda g_x = (2x+1) + \lambda \cdot 2x \\ 0 &= L_y = f_y + \lambda g_y = 4y + \lambda \cdot 2y \\ 0 &= L_\lambda = g(x,y) = x^2+y^2-1 \end{aligned}\right\} \rightarrow 2y(2+\lambda) = 0$$

$y = 0$: $x^2 = 1 - y^2 = 1$, $x = \mp 1$

$\lambda = -2$: $2x+1-4x = 0$, $x = \frac{1}{2}$, $y^2 = 1 - x^2 = \frac{3}{4}$, $y = \mp \frac{\sqrt{3}}{2}$

$f(-\frac{1}{2}, 0) = -\frac{1}{4}$ → the abs. min value

$f(-1, 0) = 0$

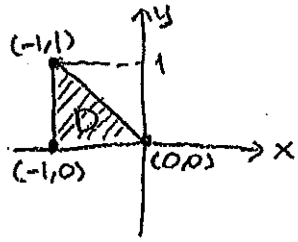
$f(1, 0) = 2$

$f(\frac{1}{2}, \mp\frac{\sqrt{3}}{2}) = \frac{9}{4}$ → the abs. max value.

Answer

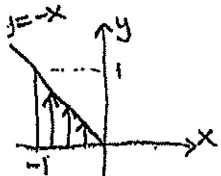
abs. max value: $\frac{9}{4}$
abs. min value: $-\frac{1}{4}$

Q.5 (14 pts) Find the volume of the solid between the surface $z = x^2 + y^2 - 2$ and the plane $z = 0$ over the triangular region in the xy-plane with vertices $(0,0), (-1,1), (-1,0)$.



$z = x^2+y^2-2 \le 0$ on D

$V = \text{Volume} = -\iint_D (x^2+y^2-2)\,dA.$

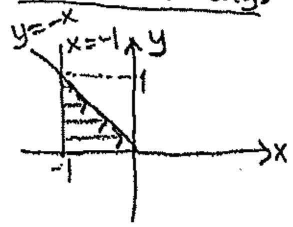


$$\Rightarrow V = \int_{-1}^{0}\int_{0}^{-x} (2-x^2-y^2)\,dy\,dx = \int_{-1}^{0}\left(2y - x^2y - \frac{y^3}{3}\right)\Big|_{y=0}^{y=-x} dx$$

$$= \int_{-1}^{0}\left(\left(-2x + x^3 + \frac{x^3}{3}\right) - 0\right) dx = \left(-x^2 + \frac{x^4}{4} + \frac{x^4}{12}\right)\Big|_{-1}^{0}$$

$$= 0 - \left(-1 + \frac{1}{4} + \frac{1}{12}\right) = \frac{2}{3} \text{ cubic units}$$

Alternative way:



$$V = \int_{0}^{1}\int_{-1}^{-y} (2-x^2-y^2)\,dx\,dy = \int_{0}^{1}\left(2x - \frac{x^3}{3} - y^2x\right)\Big|_{x=-1}^{x=-y} dy$$

$$= \int_{0}^{1}\left(\left(-2y + \frac{y^3}{3} + y^3\right) - \left(-2 + \frac{1}{3} + y^2\right)\right) dy$$

$$= \int_{0}^{1}\left(-2y + \frac{4}{3}y^3 + \frac{5}{3} - y^2\right) dy = \left(-y^2 + \frac{y^4}{3} + \frac{5y}{3} - \frac{y^3}{3}\right)\Big|_{0}^{1}$$

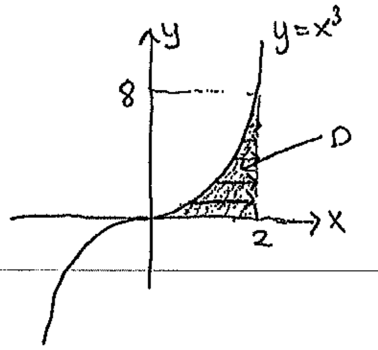
$$= \left(-1 + \frac{1}{3} + \frac{5}{3} - \frac{1}{3}\right) - 0 = 2/3 \text{ cubic units.}$$

Answer

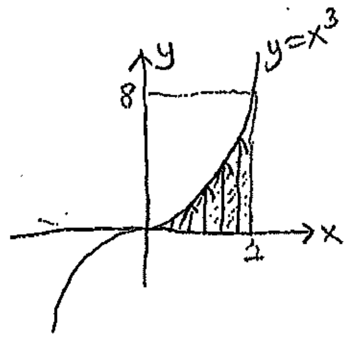
$\boxed{\frac{2}{3} \text{ cubic units}}$

Q.6 (14 pts) Evaluate the iterated integral $\int_0^8 \int_{y^{1/3}}^2 [1 + \cos(x^4)]\,dx\,dy$.

(Hint: First draw the region of integration.)



$$\int_0^8 \int_{y^{1/3}}^{2} (1+\cos(x^4))\,dx\,dy = \iint_D (1+\cos(x^4))\,dA$$



$$\iint_D (1+\cos(x^4))\,dA = \int_0^2 \int_0^{x^3} (1+\cos(x^4))\,dy\,dx$$

$$= \int_0^2 (1+\cos(x^4))(x^3 - 0)\,dx$$

$u = x^4$

$du = 4x^3 dx$

$$= \int_0^{16} \frac{1+\cos(u)}{4}\,du = \frac{u + \sin u}{4}\Big|_0^{16}$$

$$= \frac{16 + \sin(16)}{4} - 0 = 4 + \frac{\sin(16)}{4}$$

Answer

$\boxed{4 + \frac{\sin(16)}{4}}$

MATH - 1139

final

- 2017 -

(Yaz okulu)

METU Department of Mathematics

Math 118 Calculus II Midterm III 11.08.2017 18:00		
Last Name : Name : Student No:	Signature : (FINAL) Duration : 118 *minutes*	Section :
5 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS
1 | 2 | 3 | 4 | 5	SHOW YOUR WORK	

1. (10+8 pts) Given $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

(a) Find the set of points where $f(x,y)$ is continuous.

For $(x,y) \neq (0,0)$; since $x^2+y^2 \neq 0$, $f(x,y)$ is a ratio of continuous functions ($\sin(xy)$ is continuous for $\forall x,y$).

At $(x,y)=(0,0)$: $0 = f(0,0) \stackrel{?}{=} \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

Along the line $y=mx$: $\lim_{\substack{(x,y)\to(0,0)\\(y=mx)}} \frac{\sin(xy)}{x^2+y^2} = \lim_{x\to 0} \frac{\sin(mx^2)}{x^2+m^2x^2} = \lim_{x\to 0} \frac{\sin(mx^2)}{x^2(m^2+1)} = \frac{m}{1+m^2}$

since limit depends on m, $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ does NOT exist

$\Rightarrow$ f is not continuous at $(0,0)$. Thus f is continuous on $\mathbb{R}^2 - \{(0,0)\}$

(b) Calculate $f_x(x,y)$ if it exists.

For $(x,y) \neq (0,0)$: $f_x(x,y) = \frac{y\cos(xy)(x^2+y^2) - 2x\sin(xy)}{(x^2+y^2)^2}$

At $(x,y)=(0,0)$: $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin 0}{h^2} - 0}{h} = 0$

$\Rightarrow f_x(x,y) = \begin{cases} \dfrac{y\cos(xy)(x^2+y^2) - 2x\sin(xy)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

2. (7+7 pts) Find the limit, if exists. Otherwise, show that the limit does not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^6+y^4}$

For all $(x,y) \neq (0,0)$, we have $y^4 \leq y^4 + x^6 \Rightarrow \frac{y^4}{y^4+x^6} \leq 1$.

$$0 \leq \left|\frac{xy^4}{x^6+y^4}\right| \leq |x|$$

as $(x,y)\to(0,0)$, both sides $\to 0$.

By squeeze Thm, $\lim_{(x,y)\to(0,0)} \left|\frac{xy^4}{x^6+y^4}\right| = 0$.

Since $-\left|\frac{xy^4}{x^6+y^4}\right| \leq \frac{xy^4}{x^6+y^4} \leq \left|\frac{xy^4}{x^6+y^4}\right|$

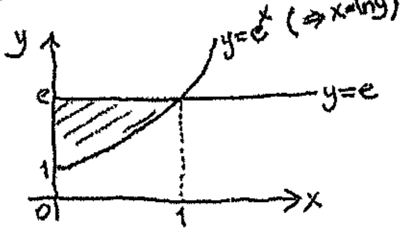
as $(x,y)\to(0,0)$, both sides $\to 0$.

By squeeze Thm, $\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^6+y^4} = 0$.

(b) $\lim_{(x,y)\to(0,0)} \frac{1-e^{(x^2+y^2-1)}}{x^2+y^2-1} = \frac{1-e^{-1}}{-1} = \frac{1}{e} - 1$

3. (6+8 pts) Let $I = \iint_D f(x,y)\,dA = \int_0^1 \int_{e^x}^{e} f(x,y)\,dy\,dx$ where $f(x,y) = \frac{y}{\ln(y)}$.

(a) Sketch the region D.



$D = \{(x,y) : e^x \leq y \leq e,\ 0 \leq x \leq 1\}$

(b) Rewrite I in $dx\,dy$ order and evaluate it.

$$I = \int_1^e \int_0^{\ln y} \frac{y}{\ln(y)}\,dx\,dy = \int_1^e \frac{y}{\ln y} \ln y\,dy = \left[\frac{y^2}{2}\right]_1^e = \frac{e^2-1}{2}.$$

4. (6+8+8+8 pts) Let $f(s,t)$ be a function with $f(-1,3)=0$, $f_1(-1,3)=2$ and $f_2(-1,3)=-3$.

(a) Find directional derivative of $f(s,t)$ at the point $(s,t)=(-1,3)$ in the direction of vector $\vec{u}=\langle 3,4\rangle$.

$\vec{u} = \langle 3,4\rangle$ is NOT unit $\Rightarrow \frac{\vec{u}}{|\vec{u}|} = \langle \frac{3}{5}, \frac{4}{5}\rangle$ is unit (as $|\vec{u}|=5$)

$$\Rightarrow D_{\vec{u}} f(-1,3) = \nabla f(-1,3)\cdot\frac{\vec{u}}{|\vec{u}|} = \langle f_1(-1,3), f_2(-1,3)\rangle\cdot\langle \tfrac{3}{5},\tfrac{4}{5}\rangle = \langle 2,-3\rangle\cdot\langle\tfrac{3}{5},\tfrac{4}{5}\rangle = -6/5$$

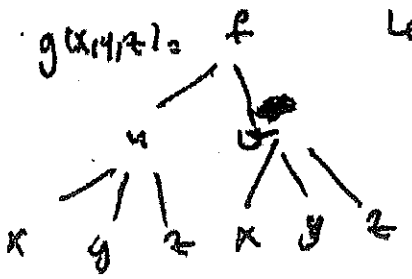
(b) Find a linear approximation for $f(s,t)$ to approximate $f(-1.05, 2.97)$.

$f(s,t) \approx L(s,t) = f(a,b) + f_1(a,b)(s-a) + f_2(a,b)(t-b)$ where $s=-1.05$, $t=2.97$, $a=-1$, $b=3$, $f(-1,3)=0$, $f_1(-1,3)=2$, $f_2(-1,3)=-3$

$\Rightarrow f(-1.05, 2.97) \cong 0 + 2\cdot(-0.05) - 3(-0.03)$

$\simeq -0.1 + 0.09 = -0.01$

(c) Let $g(x,y,z) = f(xyz, x^2+y^2+z^2)$. Find the gradient vector $\nabla g(1,-1,1)$.



Let $u = xyz$, $v = x^2+y^2+z^2$

At $(x,y,z) = (1,-1,1) \Rightarrow (u,v) = (-1,3)$

$\Rightarrow \nabla g(1,-1,1) = \langle g_x(1,-1,1), g_y(1,-1,1), g_z(1,-1,1)\rangle$

$g_x = \frac{\partial g}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \Rightarrow g_x(1,-1,1) = f_1(-1,3)\cdot(-1) + f_2(-1,3)\cdot 2 = -8$

$g_y = \frac{\partial g}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \Rightarrow g_y(1,-1,1) = f_1(-1,3)\cdot 1 + f_2(-1,3)\cdot(-2) = 8$

$g_z = \frac{\partial g}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} \Rightarrow g_z(1,-1,1) = f_1(-1,3)(-1) + f_2(-1,3)\cdot 2 = -8$

$\Rightarrow \nabla g(1,-1,1) = \langle -8, 8, -8\rangle$

(d) Find an equation of tangent plane to the surface $f(xyz, x^2+y^2+z^2)=0$ at the point $(x,y,z)=(1,-1,1)$.

$\vec{n} \parallel \nabla g = \langle -8, 8, -8\rangle$

Tangent Plane equation: $-8(x-1) + 8(y+1) - 8(z-1) = 0$

$\Rightarrow x - y + z = 3$

5. (10+10+4 pts) Let $f(x,y)=e^{xy}$ and $D=\{(x,y): 2x^2+y^2\le 1\}$.

(a) Find and classify all critical points of $f(x,y)$ in the region $\{(x,y): 2x^2+y^2<1\}$.

$$f_x = ye^{xy} \Rightarrow f_x=0 \Leftrightarrow \boxed{y=0}$$
$$f_y = xe^{xy} \Rightarrow f_y=0 \Leftrightarrow \boxed{x=0}$$
$$\nabla f = \langle f_x, f_y\rangle = 0 \Leftrightarrow x=0,\ y=0$$

(0,0) is the only CP

$$A=f_{xx}=y^2e^{xy}$$
$$B=f_{xy}=e^{xy}+xye^{xy}$$
$$C=f_{yy}=x^2e^{xy}$$

At (0,0) $B^2-AC=1>0 \Rightarrow$ By Second Derivative Test, (0,0) is a saddle point.

(b) Find the extreme values of $f(x,y)$ on the boundary $\{(x,y): 2x^2+y^2=1\}$ by using Lagrange multipliers method.

$$f(x,y)=e^{xy},\quad g(x,y)=2x^2+y^2=1,\quad \nabla f=\lambda\nabla g \Rightarrow \langle ye^{xy}, xe^{xy}\rangle = \lambda\langle 4x, 2y\rangle$$

(1) $ye^{xy}=4\lambda x \Rightarrow 4\lambda = \frac{y}{x}e^{xy}$

(2) $xe^{xy}=2\lambda y \Rightarrow 2\lambda = \frac{x}{y}e^{xy}$

$\frac{2x}{y}=\frac{y}{x} \Rightarrow 2x^2=y^2 \Leftrightarrow \boxed{2x^2-y^2=0}$ (4)

(3) $2x^2+y^2=1$

By (3) and (4) $\Rightarrow 4x^2=1 \Leftrightarrow x=\pm\frac{1}{2}$

$$\boxed{\begin{array}{l} x=\frac{1}{2} \Rightarrow y=\pm\frac{1}{\sqrt2} \\ x=-\frac{1}{2} \Rightarrow y=\pm\frac{1}{\sqrt2}\end{array}}$$

$\left(\frac12,\frac{-1}{\sqrt2}\right),\left(\frac12,\frac{1}{\sqrt2}\right)$

$\left(\frac{-1}{2},\frac{1}{\sqrt2}\right),\left(\frac{-1}{2},\frac{-1}{\sqrt2}\right)$

- $f\left(\frac12,\frac{-1}{\sqrt2}\right)=f\left(-\frac12,\frac{1}{\sqrt2}\right)=e^{\frac{-1}{2\sqrt2}}$ "the minimum of f on the boundary"
- $f\left(\frac12,\frac{1}{\sqrt2}\right)=f\left(\frac{-1}{2},\frac{-1}{\sqrt2}\right)=e^{\frac{1}{2\sqrt2}}$ "the maximum of f on the boundary"

(c) Explain why does $f(x,y)$ have absolute maximum and minimum on the region D? Find these values.

Since f(x,y) is continuous on the closed bounded region D, by Extreme Value Theorem, it has absolute maximum and minimum on D.

$f\left(\frac12,\frac{1}{\sqrt2}\right)=f\left(\frac{-1}{2},\frac{-1}{\sqrt2}\right)=e^{\frac{1}{2\sqrt2}}$ absolute maximum value

$f\left(\frac{-1}{2},\frac{1}{\sqrt2}\right)=f\left(\frac12,\frac{-1}{\sqrt2}\right)=e^{\frac{-1}{2\sqrt2}}$ absolute minimum value

Declaration of Honesty: By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

Signature : ....

P35

MATH-118

final

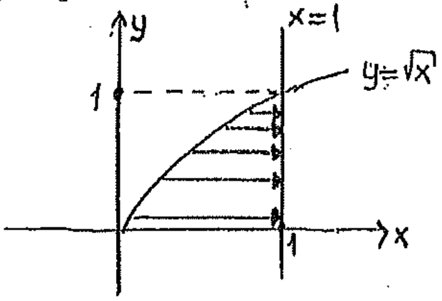
-2017

M E T U
Mathematics Department

Math 118 Calculus II Spring 2017 FINAL EXAM						
Küçüksakallı, Önsiper, Tezer. June 01, 2017 9:30 – 11:45	Name :			Student Number :		
	Last Name :			Signature :		
Q.1 25	Q.2 25	Q.3 25	Q.4 25	SHOW YOUR ORGANIZED WORK	Total 100	
				NO CALCULATORS		

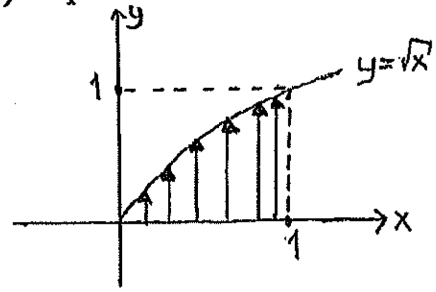
Q.1) Consider the solid lying above the region $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le \sqrt{x}\}$ and below the surface $z = f(x,y) = \dfrac{x^2 y}{1+x^4}$.

a) Express its volume as an iterated integral in the form $\int_?^? \left(\int_?^? f(x,y)dx\right) dy$.



$$\text{volume} = \int_0^1 \int_{y^2}^1 f(x,y)\, dx\, dy$$

b) Express its volume as an iterated integral in the form $\int_?^? \left(\int_?^? f(x,y)dy\right) dx$.



$$\text{volume} = \int_0^1 \int_0^{\sqrt{x}} f(x,y)\, dy\, dx$$

c) Compute its volume by using one of the iterated integrals you obtained in the previous parts.

$$\text{volume} = \int_0^1 \int_0^{\sqrt{x}} \frac{x^2 y}{1+x^4}\, dy\, dx = \int_0^1 \frac{x^2 y^2}{2(1+x^4)} \Bigg|_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \frac{x^3}{2(1+x^4)}\, dx \qquad u = 1+x^4, \quad du = 4x^3 dx$$

$$= \frac{1}{8}\int_1^2 \frac{du}{u} = \frac{1}{8} \ln u \Big|_1^2 = \frac{1}{8}\ln 2 \text{ cubic units}$$

Q.2) Let $f(x,y) = e^{-xy}$.

a) Determine and classify the critical points of $f(x,y)$ in the open disc

$$D = \{(x,y) : x^2 + y^2 < 1\}.$$

$$\left.\begin{array}{l} f_x = -y e^{-xy} = 0 \\ f_y = -x e^{-xy} = 0 \end{array}\right\} \Rightarrow x=y=0 \Rightarrow \text{The only critical point is } (0,0).$$

$A = f_{xx} = y^2 e^{-xy}$

$B = f_{xy} = -e^{xy} + xy e^{xy}$

$C = f_{yy} = x^2 e^{-xy}$

At $(0,0)$: $A = 0$, $B = -1$, $C = 0$

$B^2 - AC = 1 > 0$

$(0,0)$ is a saddle point.

b) Use the method of Lagrange multipliers to determine the extreme values of $f(x,y)$ on the circle

$$C = \{(x,y) : x^2 + y^2 = 1\}.$$

Let $g(x,y) = x^2+y^2-1$.

$$\left\{\begin{array}{l} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} -y e^{-xy} = \lambda 2x \\ -x e^{xy} = \lambda 2y \\ x^2+y^2-1 = 0 \end{array}\right.$$

$$\Rightarrow 2\lambda x^2 = -xy e^{-xy} = 2\lambda y^2 \Rightarrow 2\lambda(x^2-y^2) = 0 \Rightarrow$$

$\lambda = 0 \Rightarrow x=y=0$ ↯

or

$x^2-y^2=0 \Rightarrow 2x^2-1=0$

$x = \mp\frac{1}{\sqrt{2}}$

$y = \mp\frac{1}{\sqrt{2}}$

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = e^{-1/2} \to$ the min. value

$f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = e^{1/2} \to$ the max value

c) Let R be the region given by $R = \{(x,y) : x^2 + y^2 \leq 1\}$. Does $f(x,y)$ have an absolute maximum or minimum in R? Why?

The function f is continuous on the closed and bounded region R.

Thus f has an absolute max and an absolute min on R.

Q.3) Consider the function $f(x,y) = x^2y$.

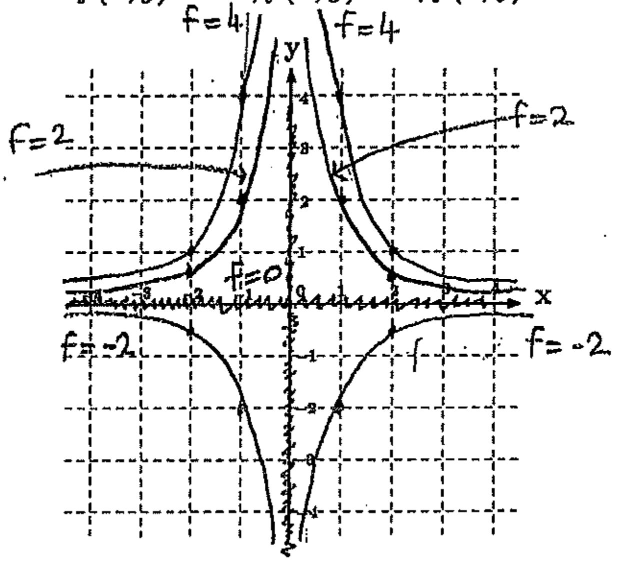
a) Sketch the level curves $f(x,y) = -2$, $f(x,y) = 0$, $f(x,y) = 2$ and $f(x,y) = 4$.

$f=-2 \Rightarrow y = \frac{-2}{x^2}$

$f=0 \Rightarrow x=0$ or $y=0$

$f=2 \Rightarrow y = \frac{2}{x^2}$

$f=4 \Rightarrow y = \frac{4}{x^2}$



b) Find the directional derivative of the function $f(x,y)$ at the point $(1,2)$ in the direction of $\mathbf{u} = -3\mathbf{i} - 4\mathbf{j}$.

$$D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \frac{\vec{u}}{|\vec{u}|} = \langle 2xy, x^2 \rangle \Big|_{\substack{x=1 \\ y=2}} \cdot \frac{\langle -3,-4 \rangle}{5}$$

$$= \langle 4,1 \rangle \cdot \frac{\langle -3,-4 \rangle}{5} = \frac{-12-4}{5} = \frac{-16}{5}$$

c) In what directions at the point $(1,2)$, is the rate of change of $f(x,y)$ maximum? Find this maximum rate of change.

Let $\vec{u}$ be any vector.

$$D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \frac{\vec{u}}{|\vec{u}|} = |\vec{\nabla} f(1,2)| \cos\theta \text{ is max when } \cos\theta = 1 \Rightarrow \theta = 0.$$

The rate of change at $(1,2)$ is maximum in the direction of $\vec{\nabla} f(1,2) = \langle 4,1 \rangle$.

The maximum rate of change at $(1,2)$ is $|\vec{\nabla} f(1,2)| = |\langle 4,1 \rangle| = \sqrt{17}$.

d) In what directions at the point $(1,2)$, is the rate of change of $f(x,y)$ zero? Why?

Let $\vec{u} = \langle a,b \rangle$ be a unit vector so that $a^2+b^2=1$.

$$D_{\vec{u}} f(1,2) = \vec{\nabla} f(1,2) \cdot \vec{u} = \langle 4,1 \rangle \cdot \langle a,b \rangle = 4a+b$$

$$D_{\vec{u}} f(1,2) = 0 \Rightarrow 4a+b=0 \Rightarrow a^2+(-4a)^2=1 \Rightarrow \begin{cases} a = \frac{1}{\sqrt{17}} \\ b = \frac{-4}{\sqrt{17}} \end{cases} \text{ OR } \begin{cases} a = \frac{-1}{\sqrt{17}} \\ b = \frac{4}{\sqrt{17}} \end{cases}$$

The directions are $\langle \frac{1}{\sqrt{17}}, \frac{-4}{\sqrt{17}} \rangle$ and $\langle \frac{-1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle$.

Q.4) Evaluate the following integrals.

a) $\displaystyle\int \frac{x}{x^2+2x-8}\,dx$ $\quad \displaystyle\frac{x}{x^2+2x-8} = \frac{A}{x+4} + \frac{B}{x-2} \Rightarrow \left.\begin{array}{l} A+B=1 \\ -2A+4B=0 \end{array}\right\} \Rightarrow \begin{array}{l} B=\frac{1}{3} \\ A=\frac{2}{3} \end{array}$

$$\int \frac{x}{x^2+2x-8}\,dx = \int\left(\frac{2}{3(x+4)} + \frac{1}{3(x-2)}\right)dx = \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$$

b) $\displaystyle\int x^2 2^x\,dx \overset{(*)}{=} \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2}\int x 2^x\,dx \overset{(**)}{=} \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2}\left(\frac{x2^x}{\ln 2} - \int \frac{2^x}{\ln 2}\,dx\right)$

$$= \frac{x^2 2^x}{\ln 2} - \frac{x\,2^{x+1}}{(\ln 2)^2} + \frac{2^{x+1}}{(\ln 2)^3} + C$$

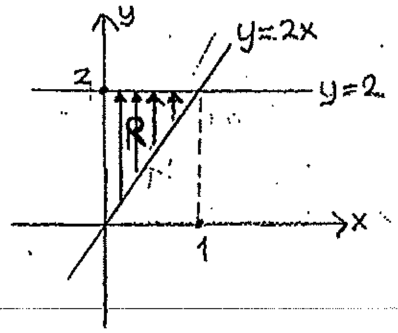
$(*)$

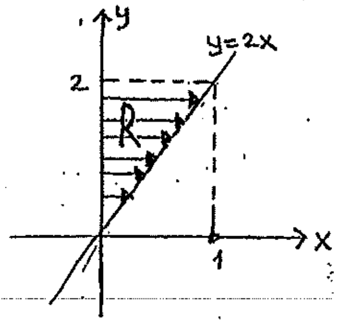
$$\boxed{\begin{array}{ll} u = x^2 & dv = 2^x\,dx \\ du = 2x\,dx & v = \frac{2^x}{\ln 2} \end{array}}$$

$(**)$

$$\boxed{\begin{array}{ll} u = x & dv = 2^x\,dx \\ du = dx & v = \frac{2^x}{\ln 2} \end{array}}$$

c) $\displaystyle\int_0^1\left(\int_{2x}^2 \cos(y^2)\,dy\right)dx = \iint_R \cos(y^2)\,dA = \int_0^2\int_0^{y/2} \cos(y^2)\,dx\,dy$

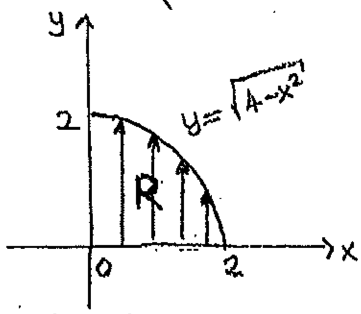


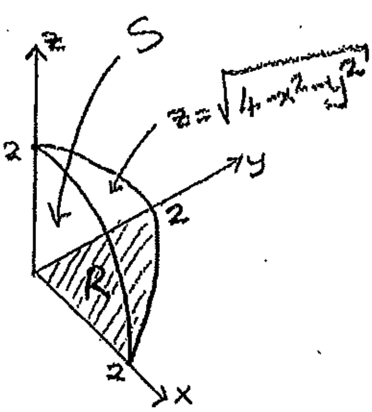


$$= \int_0^2 \frac{y}{2}\cos(y^2)\,dy$$

$$= \left.\frac{\sin(y^2)}{4}\right|_0^2 = \frac{\sin 4}{4}$$

d) $\displaystyle\int_0^2\left(\int_0^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\,dy\right)dx$





$$\int_0^2\int_0^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\,dy\,dx = \iint_R \sqrt{4-x^2-y^2}\,dA = \text{volume}(S) = \frac{1}{8}\left(\frac{4}{3}\pi 2^3\right) = \frac{4\pi}{3}$$

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