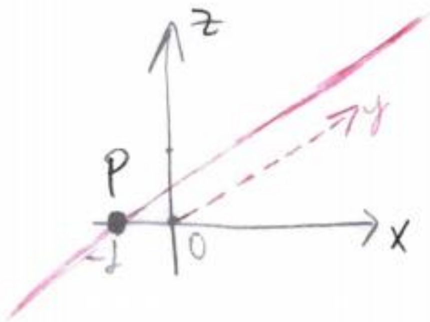
Exercises 12.1

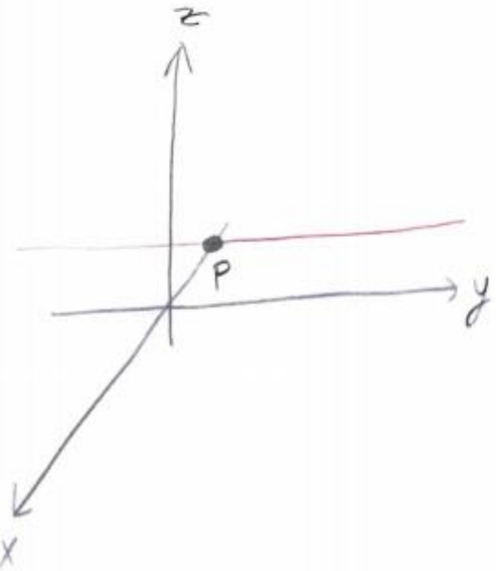
Geometric Interpretations of Equations

In Exercises 1–16, give a geometric description of the set of points in space whose coordinates satisfy the given pairs of equations.

2. $x = -1, \quad z = 0$

$(-1, y, 0)$ y is any real number. This is a line.

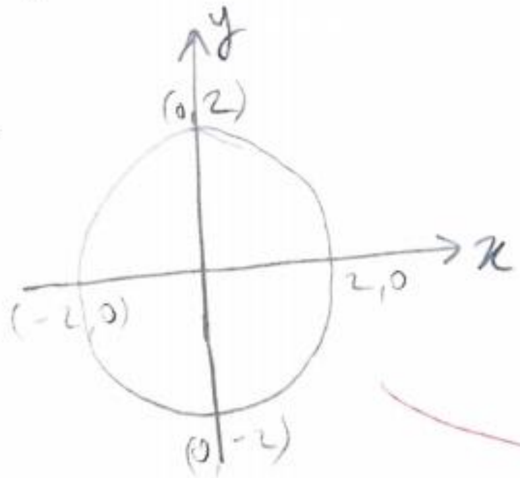


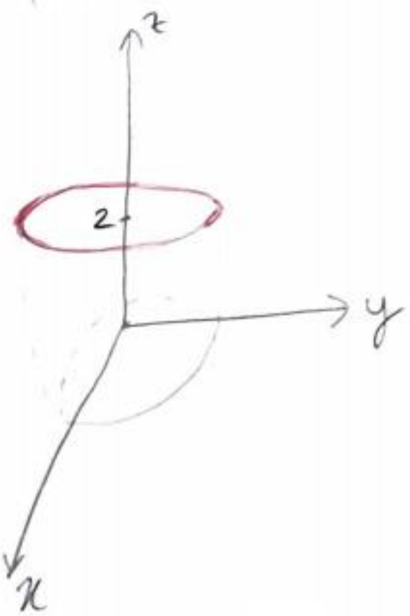


6. $x^2 + y^2 = 4, \quad z = -2$

$x^2+y^2=2^2$ is a circle on xy-plane

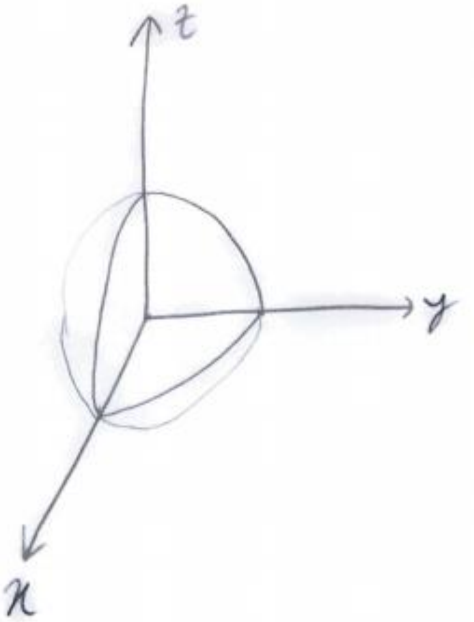
$z=-2$ means we push this circle up.

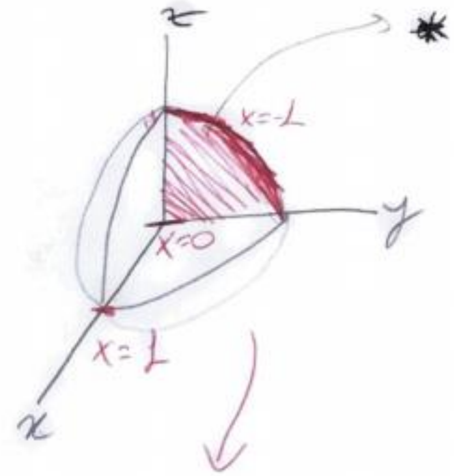


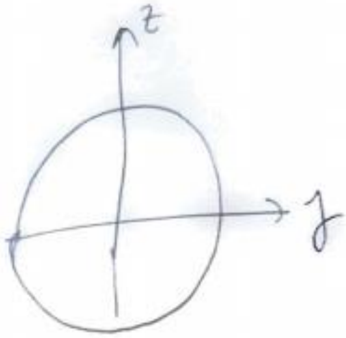


9. $x^2 + y^2 + z^2 = 1, \quad x = 0$

$x^2+y^2+z^2=1$ is a sphere in $\mathbb{R}^3$. However, we select a piece of this sphere.







OR

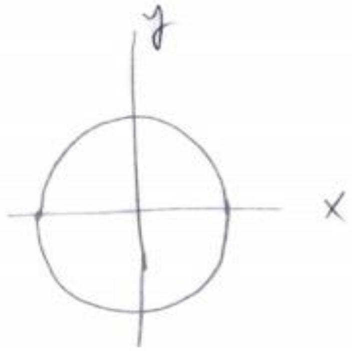
by putting $x=0$, we get

$y^2+z^2=1$ a circle.
in yz-plane.

* → We are not interested in that shaded area.

13. $x^2 + y^2 = 4, \quad z = y$

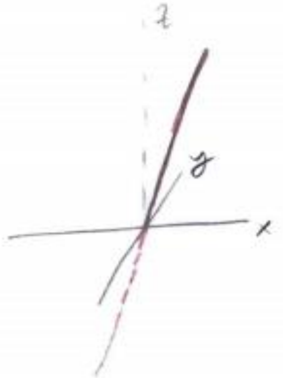
$x^2+y^2=4$ is a circle in xy plane!

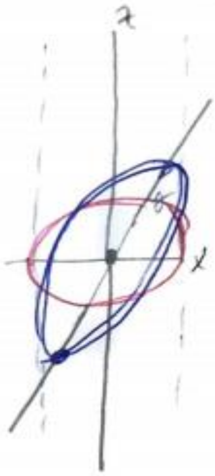


This is projection of the wanted shape on the xy plane. So it can be

We should look at the properties of 3rd component.

$z = y$





OR

$x^2+y^2=4$ is a cylinder in $\mathbb{R}^3$

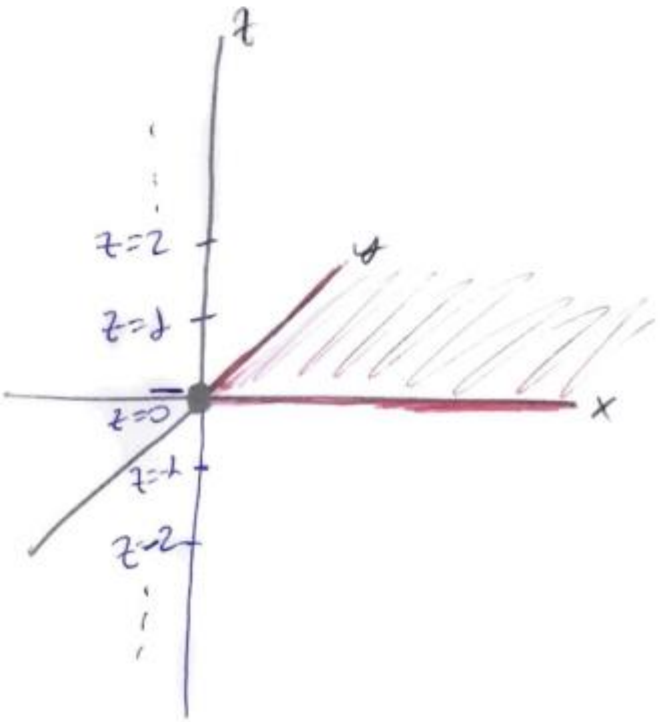
$y=z$ plane

the intersection is

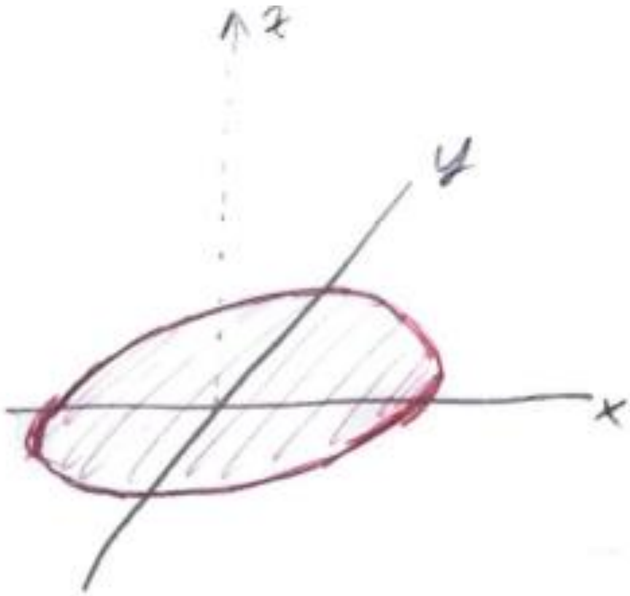
Geometric Interpretations of Inequalities and Equations

In Exercises 17–24, describe the sets of points in space whose coordinates satisfy the given inequalities or combinations of equations and inequalities.

17. **a.** $x \geq 0, \quad y \geq 0, \quad z = 0$



20. **a.** $x^2 + y^2 \leq 1, \quad z = 0$



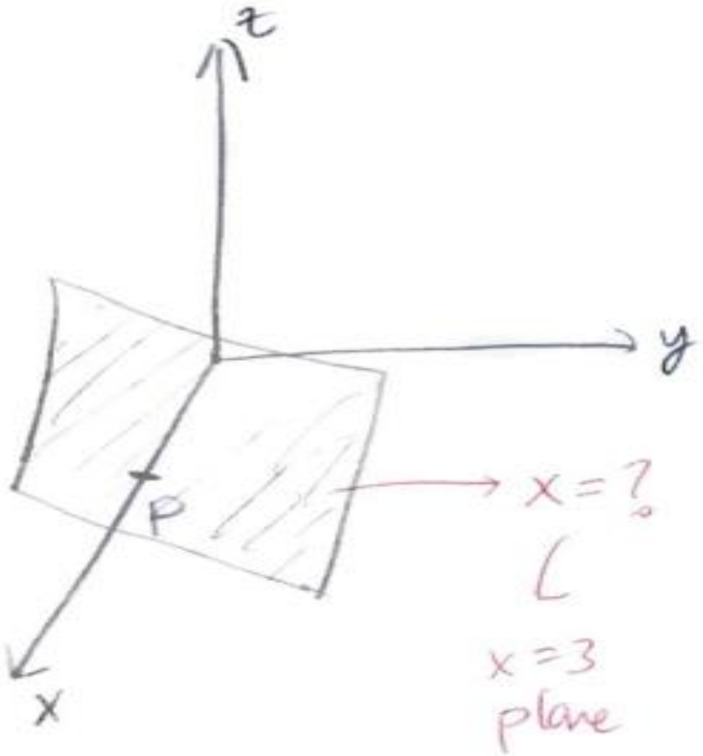
In Exercises 25–34, describe the given set with a single equation or with a pair of equations.

25. The plane perpendicular to the

a. x-axis at $(3, 0, 0)$ **b.** y-axis at $(0, -1, 0)$

c. z-axis at $(0, 0, -2)$

$P = (3,0,0)$



30. The circle of radius 1 centered at $(-3, 4, 1)$ and lying in a plane parallel to the

a. xy-plane **b.** yz-plane **c.** xz-plane

to parallel $xy \rightarrow z$ must be constant

to parallel $yz \rightarrow x$ must be constant

to parallel $xz \rightarrow y$ must be constant

$\Rightarrow$

a. $z = 1$ $\Downarrow$ $(x+3)^2 + (y-4)^2 = 1$

b. $x = -3$ $\Downarrow$ $(y-4)^2 + (z-1)^2 = 1$

c. $y = 4$ $\Downarrow$ $(x+3)^2 + (z-1)^2 = 1$

33. The circle in which the plane through the point (1, 1, 3) perpendicular to the z-axis meets the sphere of radius 5 centered at the origin

perpendicular to z-axis $\Rightarrow$ $z = b$ plane

passing through $(1,1,3) \Rightarrow \boxed{z=3}$

The answer must be $x^2+y^2=R^2$ and $z=3$.

$x^2+y^2+z^2=25 \Rightarrow x^2+y^2+9=25 \Rightarrow x^2+y^2=16$

The answer $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=16 \text{ and } z=3\}$

Distance

In Exercises 41–46, find the distance between points P_1 and P_2.

43. $P_1(1, 4, 5)$, $\quad P_2(4, -2, 7)$

$$d(P_1,P_2)=\sqrt{(1-4)^2+(4-(-2))^2+(5-7)^2}=\sqrt{9+36+4}=\sqrt{49}=\boxed{7}$$

Spheres

Find the centers and radii of the spheres in Exercises 47–50.

49. $(x-\sqrt{2})^2+(y-\sqrt{2})^2+(z+\sqrt{2})^2=2$

Center $\rightarrow (\sqrt{2}, \sqrt{2}, -\sqrt{2})$ radii $\rightarrow \sqrt{2}$

Find the centers and radii of the spheres in Exercises 55–58.

57. $2x^2 + 2y^2 + 2z^2 + x + y + z = 9$

$$x^2+y^2+z^2+\frac{x}{2}+\frac{y}{2}+\frac{z}{2}=\frac{9}{2}$$

$$x^2+\frac{x}{2}=\left(x+\frac{1}{4}\right)^2-\frac{1}{16}$$

$$\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\frac{9}{2}+\frac{3}{16}$$

$$=\frac{75}{16}$$

Center $\rightarrow \left(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$ radii $\rightarrow \frac{5\sqrt{3}}{4}$

Theory and Examples

64. Find an equation for the set of all points equidistant from the point (0, 0, 2) and the xy-plane.

xy plane $\rightarrow z=0$

let us select a point, let it be $(x,y,z)=P$

$$d(P, xy\text{-plane}) = z$$

$$d(P, (0,0,2)) = \sqrt{x^2+y^2+(z-2)^2}$$

they are equal.

$$z=\sqrt{x^2+y^2+(z-2)^2} \Rightarrow z^2=x^2+y^2+(z-2)^2$$

$$\Rightarrow 4z-4=x^2+y^2 \Rightarrow \boxed{z=\frac{x^2}{4}+\frac{y^2}{4}+1}$$

Exercises 12.2

Vectors in the Plane

In Exercises 1–8, let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Find the (a) component form and (b) magnitude (length) of the vector.

5. $2\mathbf{u} - 3\mathbf{v}$

$$\mathbf{u} = \langle 3, -2 \rangle \qquad 2\mathbf{u} - 3\mathbf{v} = 2\langle 3, -2 \rangle - 3\langle -2, 5 \rangle$$

$$\mathbf{v} = \langle -2, 5 \rangle$$

$$\begin{aligned} 2\mathbf{u} - 3\mathbf{v} &= \langle 2(3), 2(-2) \rangle + \langle -3(-2), -3(5) \rangle \\ &= \langle 6, -4 \rangle + \langle 6, -15 \rangle \\ &= \langle 6 + 6, -4 + (-15) \rangle \\ &= \langle 12, -19 \rangle \end{aligned}$$

$$|2\mathbf{u} - 3\mathbf{v}| = |\langle 12, -19 \rangle|$$

$$|2\mathbf{u} - 3\mathbf{v}| = \sqrt{12^2 + (-19)^2}$$

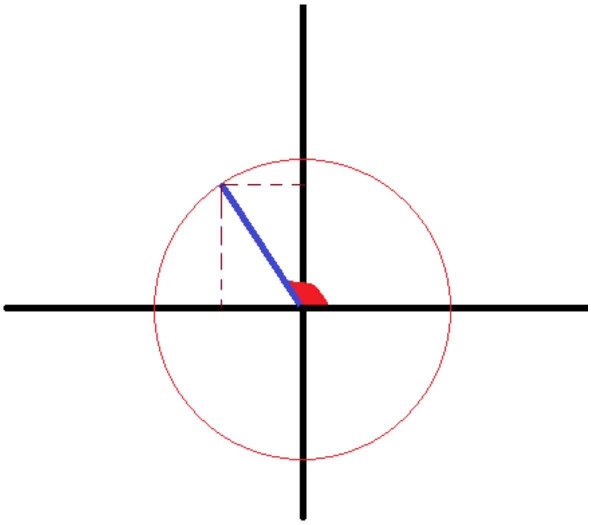
$$|2\mathbf{u} - 3\mathbf{v}| = \sqrt{144 + 361} = \sqrt{505}$$

In Exercises 9–16, find the component form of the vector.

9. The vector $\overrightarrow{PQ}$, where $P = (1, 3)$ and $Q = (2, -1)$

$$\overrightarrow{QP} = Q - P = (2, -1) - (1, 3) = \langle 1, -4 \rangle$$

13. The unit vector that makes an angle $\theta = 2\pi/3$ with the positive x-axis



$(cos\theta, sin\theta)$ is any element in the unit circle.

$$(cos\tfrac{2\pi}{3}, sin\tfrac{2\pi}{3}) = (-1/2, \sqrt{3}/2)$$

Vectors in Space

In Exercises 17–22, express each vector in the form $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

21. $5\mathbf{u} - \mathbf{v}$ if $\mathbf{u} = \langle 1, 1, -1 \rangle$ and $\mathbf{v} = \langle 2, 0, 3 \rangle$

$$\begin{aligned} 5u - v &= 5\langle 1,1,-1\rangle + (-1)\langle 2,0,3\rangle \\ &= \langle 5,5,-5\rangle + \langle -2,0,-3\rangle \\ &= \langle 3,5,-8\rangle \end{aligned}$$

3i+5j-8k

Length and Direction

In Exercises 25–30, express each vector as a product of its length and direction.

26. $9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$

Here the aim is to write any arbitrary vector v in the form of $v = au$ where a is a constant and u is a unit vector. So

$$v = |v|\frac{v}{|v|} \to a = |v| \qquad u = \frac{v}{|v|}$$

$a = |\langle 9, -2, 6\rangle| = 81 + 4 + 36 = 11$ and $u = \frac{\langle 9,-2,6\rangle}{11} = \frac{9}{11}i - \frac{2}{11}j + \frac{6}{11}k$

31. Find the vectors whose lengths and directions are given. Try to do the calculations without writing.

Length	Direction	
a. 2	$\mathbf{i}$	$2i$
b. $\sqrt{3}$	$-\mathbf{k}$	$-\sqrt{3}k$
c. $\frac{1}{2}$	$\frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$	$\langle 0, \frac{3}{10}, \frac{4}{10} \rangle$
d. 7	$\frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}$	$\langle 6, -2, 3 \rangle$

Exercises 12.3

Dot Product and Projections

In Exercises 1–8, find

a. $\mathbf{v} \cdot \mathbf{u}$, $|\mathbf{v}|$, $|\mathbf{u}|$

b. the cosine of the angle between $\mathbf{v}$ and $\mathbf{u}$

c. the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$

d. the vector $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

3. $\mathbf{v} = 10\mathbf{i} + 11\mathbf{j} - 2\mathbf{k}$, $\quad \mathbf{u} = 3\mathbf{j} + 4\mathbf{k}$

8. $\mathbf{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}} \right\rangle$, $\quad \mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \right\rangle$

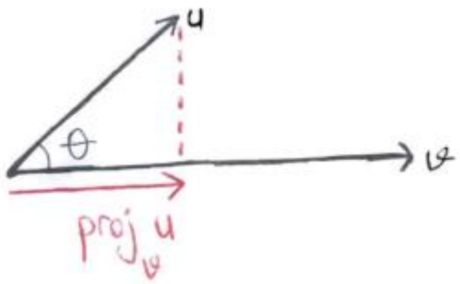
$u \cdot v = v \cdot u$

$\boxed{3}$ $\quad v \cdot u = 10.0 + 11.3 + (-2).4 = 33 - 8 = 25$

$|v| = \sqrt{v.v} = \sqrt{100 + 121 + 4} = \sqrt{225} = 15$

$|u| = \sqrt{u.u} = \sqrt{0 + 9 + 16} = 5$

$\cos\theta = \dfrac{u.v}{|u||v|} = \dfrac{25}{15.5} = \dfrac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$



Notice that $\text{proj}_v u = cv$ for some constant c. However, it is good idea to write

$\text{proj}_v u = |\text{proj}_v u| \dfrac{v}{|v|}$

$|\text{proj}_v u| = |u| \cos\theta = 5 \cdot \frac{1}{3} = \frac{5}{3}$

$\dfrac{v}{|v|} = \dfrac{\langle 10, 11, -2 \rangle}{15}$

$\Rightarrow \text{proj}_v u = \left\langle \frac{10}{9}, \frac{11}{9}, \frac{-2}{9} \right\rangle$

$\boxed{8}$ $u \cdot v = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$

$|u| = \sqrt{u \cdot u} = \sqrt{\frac{1}{2} + \frac{1}{3}} = \sqrt{\frac{5}{6}}$

$|v| = \sqrt{\frac{5}{6}}$

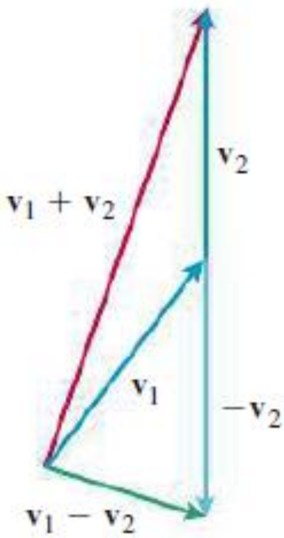
$\cos\theta = \frac{u \cdot v}{|u| \cdot |v|} = \frac{1/6}{5/6} = \frac{1}{5} \qquad \theta = \cos^{-1}\left(\frac{1}{5}\right)$

$\text{proj}_v u = |u| \cdot \cos\theta \cdot \frac{v}{|v|} = \sqrt{\frac{5}{6}} \cdot \frac{1}{5} \cdot \frac{\langle 1/\sqrt{2}, 1/\sqrt{3} \rangle}{\sqrt{\frac{5}{6}}} = \left\langle \frac{1}{5\sqrt{2}}, \frac{1}{5\sqrt{3}} \right\rangle$

$|\text{proj}_v u| = |u| \cdot \cos\theta = \sqrt{\frac{5}{6}}\ \frac{1}{5} = \frac{1}{\sqrt{30}}$

Theory and Examples

17. Sums and differences In the accompanying figure, it looks as if $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1 - \mathbf{v}_2$ are orthogonal. Is this mere coincidence, or are there circumstances under which we may expect the sum of two vectors to be orthogonal to their difference? Give reasons for your answer.



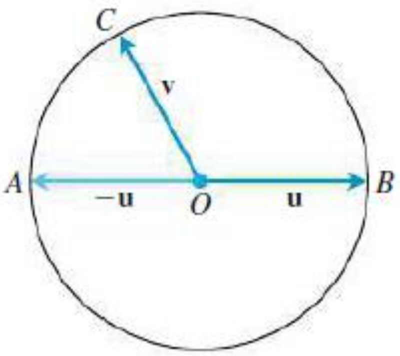
We can show x and y are orthogonal by finding

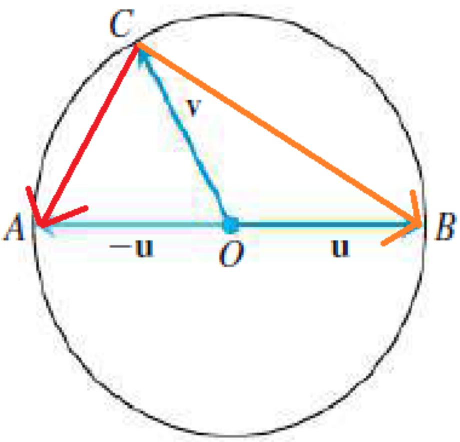
$x \cdot y = 0$

$x = v_1 + v_2 \quad y = v_1 - v_2$

$$x \cdot y = (v_1 + v_2) \cdot (v_1 - v_2)$$
$$= (v_1 + v_2) v_1 - (v_1 + v_2) v_2$$
$$= |v_1|^2 + v_1 \cdot v_2 - v_1 \cdot v_2 - |v_2|^2$$
$$= |v_1|^2 - |v_2|^2 \implies$$ x and y are orthogonal if and only if $|v_1| = |v_2|$

18. Orthogonality on a circle Suppose that AB is the diameter of a circle with center O and that C is a point on one of the two arcs joining A and B. Show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.





We have only this equation $|u| = |v|$

$\overrightarrow{CA}$:= the red arrow $\hookrightarrow \rho$

$\overrightarrow{CB}$:= the orange arrow $\hookrightarrow \omega$

$v + \rho = -u \Rightarrow \rho = -u - v \qquad v + \omega = u \Rightarrow \omega = u - v$

$\rho . \omega = (-u-v) . (u-v) = -|u|^2 + u.v - v.u + |v|^2 = 0$

$\Rightarrow$ So ρ and ω are orthogonal.

Solution of 65 in 12.1

$\boxed{a}$ Let P be any point on the sphere, let $P=(x,y,z)$ distance between P and xy-plane is $|z|$. So we want to minimize $|z|$. From the equation,

$$\underbrace{x^2}_{\geq 0}+\underbrace{(y-3)^2}_{\geq 0}+(z+5)^2=4 \Rightarrow (z+5)^2 \leq 4$$

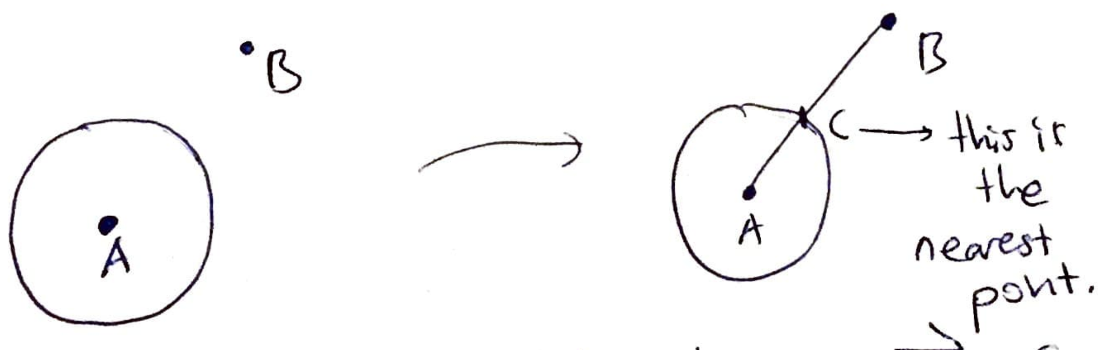
$$\Rightarrow -7 \leq z \leq -3$$

minimum value of $|z|$ is 3 when $z=3$, then

$x^2=0 \Rightarrow x=0$ and $(y-3)^2=0 \Rightarrow y=3$

Thus $P=(0,3,3)$

$\boxed{b}$ Since $0^2+(7-3)^2+(-5+5)^2=16>4$, $(0,7,-5)$ is the outside of the sphere, say B.



Let u be direction unit vector of $\overrightarrow{AB}$. So

$$C = A + \underbrace{r}_{\text{radius}}u \Rightarrow C=(0,3,-5)+(0,2,0)$$

$$\boxed{=(0,5,-5)}$$

$$u=\frac{(0,7,-5)-(0,3,-5)}{|(0,7,-5)-(0,3,-5)|}=\langle 0,1,0\rangle$$

nearest point to B.